Fall Semester '05-'06
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Lecture 2: Mathematical Preliminaries

I. OVERVIEW

- This course will require you to use basic calculus skills. The more comfortable you are with the material covered in Math 115 the easier you fill find this course to be. If it has been a while since you took calculus, I urge you to review some basic concepts.
- Since the topic at hand deals with economic growth, the basic mathematical foundations we need relate to calculating growth rates.In particular, we will learn how to do four basic tasks related to economic growth.
- First, given a) a period of time, b) the average growth rate over that period of time and c) the initial (or terminal) value of a variable, we will learn to calculate the terminal (or initial) value of that variable. In other words given a initial value and a growth rate, we can figure out what the value will be at some point in the future (or equivalently given the ending value and a growth rate, we can figure out what the initial value must have been). For this task, we make use of a special function in mathematics, known as the **exponential** function.
- Second, we will do a slight variation of that problem. Given a) a period of time b) an initial value and c) a terminal value for a variable, we will learn to calculate the average growth rate of that variable over the period. In other words given a initial value and an ending value we can figure out how fast the variable grew, on average, over the period. For this, we make use of another special function in macro known as the **logarithmic** function.
- Next, we will show how, given an algebraic equation that describes how the *level* of a variable changes with time, we will derive an algebraic equation for the *growth rate* of that variable at a given point in time.
- Finally, we will show how given a relationship among certain variables, say $Z = f(X, Y)$, we can calculate the growth rate of a variable Z, as a function of the growth rates of X and Y.

II. CALCULATING THE FUTURE (OR PRESENT) VALUE

Discrete Compounding

- Under discrete compounding, a variable x changes in value at distinct points during a period.
- We first consider the simplest case, in which the variable changes only once, at the end of the period.
- Suppose you are given the initial value x_0 and the per-period growth rate r of a variable x. The value of x after 1 period will be $x_1 = x_0(1 + r)$ and the value of x after 2 periods will be $x_2 = x_1(1 + r) = x_0(1 + r)^2$. Extending this, we can show that after t periods, the future (or compounded) value of x can be expressed as $x_t = x_0(1 + r)^t$.
- Similarly if we knew the value of x at time t, we can calculate the initial (or discounted) value of that variable as $x_0 = \frac{x_t}{(1+r)^t}$.

- Now let's make things a little more complicated by supposing that x was still growing at the rate of r a year but the compounding was happening k times a year. Ex: a bank account that pays interest monthly at an interest rate of 5% a year has r=0.05 and k=12.
- Now, the value of x after 1 period will be $x_1 = x_0(1+\frac{r}{k})^k$. Generalizing, we can show that after t periods, the future (or compounded) value of x can be expressed as $x_t = x_0\left(1+\frac{r}{k}\right)^{kt}$
- Similarly, if we knew the value of x at time t, we can calculate the initial (or discounted) value of that variable as $x_0 = \frac{x_t}{\left(1+\frac{r}{k}\right)^{kt}}$

The Exponential Function

- An **exponential** function is a function of the form $y = \alpha a^{\beta x}$ where a, α and β are constants. In economics, what we call the exponential function is one in which $a = e$, i.e. functions of the form $y = \alpha e^{\beta x}$, where the number 'e' is a very special number in mathematics defined as $lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, approximately equaling 2.718. Some useful rules of the exponential function to brush up on are the following

$$\begin{aligned} e^{ax} \times e^{bx} &= e^{(a+b)x} \\ e^{ax} \div e^{bx} &= e^{(a-b)x} \\ (e^{ax})^n &= e^{n(ax)} \end{aligned}$$

- The exponential function is vital in calculating the initial or terminal value of a variable whose value is changing **continuously** over time.

Continuous Compounding

- Under continuous compounding, a variable x is changing in value all the time.In other words unlike your bank account on which interest may be calculated annually or quarterly or monthly or daily, many economic variables are changing at every instant in time.
- We can think of continuous compounding as a special case of discrete compounding, where k approaches infinity. So the value, t periods from now, of a variable that is being compounded continuously at a rate r can be expressed as

$$x_t = \lim_{k\to\infty} x_0\left(1+\frac{r}{k}\right)^{kt}$$

- We can do a little bit of algebra and re-write this as

$$x_t = \lim_{\frac{k}{r}\to\infty} x_0\left[\left(1+\frac{1}{\frac{k}{r}}\right)^{\frac{k}{r}}\right]^{rt}$$

- This simplifies to

$$x_t = x_0 e^{rt}$$

- Suppose you are given the initial value x_0 and the per-period growth rate r of a continuously growing variable x. The value of x after t periods is $x_t = x_0 e^{rt}$.

- Similarly, if we knew the value of x at time t, we can calculate the initial (or discounted) value of that variable as $x_0 = x_t e^{-rt}$

Example:

- Suppose you put $15,000 in a bank account that pays a 5% annual rate of return. Calculate the value of the bank balance after 1, 3 and 5 years under discrete and continuous compounding.
- Under discrete compounding: $B_t = B_0(1+r)^t$. So $B_1 = \$15,750$, $B_3 = \$17,364.38$ and $B_5 = \$19,144.22$.
- Under continuous compounding $B_t = B_0 e^{rt}$. So $B_1 = \$15,769.07$, $B_3 = \$17,427.51$ and $B_5 = \$19,260.38$

III. CALCULATING THE GROWTH RATE OF A VARIABLE

The Logarithmic Function

- The **logarithmic** function in mathematics is the inverse of an exponential function. So the inverse of the exponential function $y = b^x$ is the logarithmic function $x = \log_b y$, also called the value of y in log base b.
- In economics, what we call the logarithmic function is typically the **natural logarithmic** function, i.e. logarithms in base 'e'. So the inverse of the function $y = e^x$ is the logarithmic function $x = \ln(y)$. Some useful rules of the logarithmic function to remember are the following:

$$\begin{aligned} \ln(xy) &= \ln(x) + \ln(y) & \ln\left(\frac{x}{y}\right) &= \ln(x) - \ln(y) \\ \ln(x^n) &= n\ln(x) & \ln\left(e^{f(x)}\right) &= f(x) \\ e^{\ln(f(x))} &= f(x) \end{aligned}$$

Using the Logarithm to Calculate Growth Rates

- The **growth rate** of a variable refers to the percentage rate of change of that variable over a given period of time. In most economic applications, especially in macroeconomics, growth rates are expressed in terms of the percentage rate of change over a year.
- So if the period of time is longer than a year, we typically express the 'average growth rate', i.e. by what % the variable has changed every year, on average, over the period. The average growth rate of GDP from 1960-1970 is the % increase in GDP per year, on average, from 1960-70.
- Given observations of a variable X at two points in time, let's say year 0 and year t, we can calculate the average annual growth rate of the variable as

$$\frac{\ln(X_t) - \ln(X_0)}{t}$$

- In other words, the growth rate can be found by taking the difference in the natural logs of the variable, divided by the elapsed number of years.

- Why does this formula work? Think about the following example. Suppose the capital stock grows at a rate g per period and that we observe the value of the capital stock at two periods in time (n periods apart), which we denote by K_0 and K_t

- The relationship between K_0 and K_t can be expressed as follows: under discrete compounding, $K_t = K_0(1+g)^t$, and under continuous compounding, $K_t = K_0 e^{gt}$.

- Taking logs of the expression we got under continuous compounding we get $\ln(K_t) = \ln(K_0) + \ln(e^{gt}) \equiv \ln(K_0) + gt$. This simplifies down to $\ln(K_t) - \ln(K_0) = gt$ or equivalently that $\frac{\ln(K_t)-\ln(K_0)}{t} = g$. The difference in natural logs divided by the elapsed number of periods gives us the exact average per-period growth rate of the capital stock.

- Now consider the case of discrete compounding. Since $K_t = K_0(1+g)^t$, $\ln(K_t) = \ln(K_0) + t\ln(1+g)$. This simplifies down to $\frac{\ln(K_t)-\ln(K_0)}{t} = \ln(1+g)$. This time, taking the difference in natural logs and dividing by the elapsed number of periods does not give us the exact per-period average growth rate of the capital stock. However, as long as g is not too large, the value of ln(1+g) is very close to g. [Try it out and see for yourself].

- So the difference in natural logs divided by the number of periods elapsed gives an exact answer for the average per-period growth rate for continuously compounded variables and an approximate answer for the average per-period growth rate for discretely compounded variables.

- Note that describing the average rate of growth over a period of many years in time does not mean that the variable always grew at the same rate. It may have grown faster than the average in some periods, slower than the average in others. Think back to the graph of U.S. Real GDP I showed in class on the first day - on average the U.S. economy grew at a rate of 3.25% a year but the graph clearly shows that in some years growth was higher and in other years growth was lower, but the average still has a lot of information for us.

IV. TIME DERIVATIVES AND GROWTH RATES

- In the models that we develop in this class, we will often have to derive an expression for the growth rate of a continuously growing variable at some particular instant in time.

- Consider an economic variable that changes continuously over time e.g. the capital stock K. Use the following short hand notation $\dot{K} = \frac{dK}{dt}$ to denote the derivative of K with respect to time. $\dot{K}$ then represents the instantaneous change in the variable, i.e. by how much K changes in the next instant of time. The expression $\frac{\dot{K}}{K}$ represents by what % K changes in the next instant of time, or equivalently the growth rate of K.

- In other words, given an algebraic expression for how a variable (K) evolves with time, we can calculate an algebraic expression for the instantaneous growth rate of that variable by taking the derivative of K with respect to time and dividing by K.

- Another very useful result is that

$$\frac{dln(K)}{dt} = \frac{dln(K)}{dK}\frac{dK}{dt} = \frac{1}{K}\frac{dK}{dt} = \frac{\dot{K}}{K}$$

- Basically, given an algebraic expression for a variable K, we can calculate an algebraic expression for the growth rate of that variable by taking the derivative of $ln(K)$ with respect to time. [So if you take the derivative of the log of K with respect to time you no longer need to divide by K to get an expression for the growth rate].
- In this class, we will typically express growth rates on a per-year basis. Therefore, the expression $\frac{\dot{K}}{K} = 0.05$ means that the the growth rate of the variable K at the current instant in time is a rate of 5% per year.

V. THE T.L.A.D. TECHNIQUE

- Occasionally, we come across economic variables that are related to one another in certain ways. If we know how these variables are related, we can find a relationship that should hold among the *growth rates* of the variables using a technique that we will term the "Take Logs and Differentiate" (T.L.A.D.) technique.
- This technique will prove enormously helpful in finding relationships between the growth rates of various variables. We can show that the following rules hold

  1. If $\frac{\dot{A}}{A} = a$ and $\frac{\dot{B}}{B} = b$ then the growth rate of $C = AB$ is $\frac{\dot{C}}{C} = a + b$. This can be seen by the following algebra:

$$\begin{aligned}\frac{\dot{C}}{C} &= \frac{d\ln(C)}{dt} \equiv \frac{d\ln(AB)}{dt} \equiv \frac{d\ln(A)}{dt} + \frac{d\ln(B)}{dt} \\ &= \frac{\dot{A}}{A} + \frac{\dot{B}}{B} = a + b\end{aligned}$$

  2. If $\frac{\dot{A}}{A} = a$ and $\frac{\dot{B}}{B} = b$ then the growth rate of $D = \frac{A}{B}$ is $\frac{\dot{D}}{D} = a - b$. This can be seen by the following algebra:

$$\begin{aligned}\frac{\dot{D}}{D} &= \frac{d\ln(D)}{dt} \equiv \frac{d\ln\left(\frac{A}{B}\right)}{dt} \equiv \frac{d\ln(A)}{dt} - \frac{d\ln(B)}{dt} \\ &= \frac{\dot{A}}{A} - \frac{\dot{B}}{B} = a - b\end{aligned}$$

  3. If $\frac{\dot{A}}{A} = a$ and α is an arbitrary constant then the growth rate of $F = A^{\alpha}$ is $\frac{\dot{F}}{F} = \alpha a$. This can be seen by the following algebra:

$$\begin{aligned}\frac{\dot{F}}{F} &= \frac{d\ln(F)}{dt} \equiv \frac{d\ln\left(A^{\alpha}\right)}{dt} \equiv \frac{\alpha d\ln(A)}{dt} \\ &= \alpha\frac{\dot{A}}{A} = \alpha a\end{aligned}$$