Fall Semester '05-'06
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Lecture 4: The Solow Diagram

I. OVERVIEW

- In the last lecture, we discussed the importance of economic growth, and the importance of models. We then derived a model of economic growth: the Solow growth model, which basically consisted of two equations: a production function and a capital accumulation equation.
- In today's lecture, we will first transform these equations into per-capita terms: in other words write the equations in terms of output and capital per-capita. This will make the model more tractable. We will then use these two equations to draw a diagram known as the Solow diagram.
- We will then use this diagram to do various "comparative statics" exercises where we analyze the impact of changes in saving rates, population growth rates, the capital stock etc. on economic growth in the short run and in the long run.

II. THE PER-CAPITA VERSION OF THE SOLOW MODEL

- In order to make working with the model a little easier, we transform the model into per-capita terms. Define the following per-capita variables $y = \frac{Y}{L}$ (output per worker) and $k = \frac{K}{L}$ (capital per worker)
- Note that every person is a worker in this model. I will use per-capita and per-worker interchangeably.)
- The production function can be written in per-capita terms as $\frac{Y}{L} = \frac{K^\alpha L^{1-\alpha}}{L} \equiv \frac{K^\alpha}{L^\alpha}$ which means that $y = k^\alpha$
- The capital accumulation equation can be written in per-capita terms as well.

$$
\begin{aligned}
k &= \frac{K}{L} \Rightarrow \frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \\
\frac{\dot{k}}{k} &= \frac{(sY - \delta K)}{K} - n \equiv \frac{(sY)}{K} - \delta - n \\
\frac{\dot{k}}{k} &= \frac{(sY/L)}{K/L} - (\delta + n) \equiv \frac{(sy)}{k} - (\delta + n) \\
\Rightarrow \dot{k} &= sy - (\delta + n)k
\end{aligned}
$$

- This equation states that net investment per worker (the change in the capital stock per worker) is the difference between gross investment per worker (sy) and the level of break-even investment per worker $(n + \delta)k$.
- The intuition underlying this equation is not too difficult to follow:

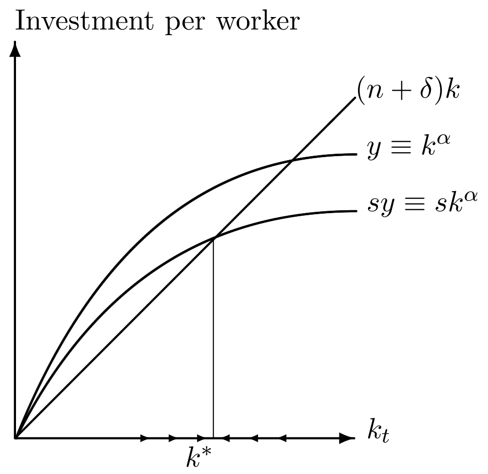
1. Gross investment per worker is equal to savings per worker (From gross investment = savings).
2. Some of this new capital per worker must be used to replenish depreciated capital per worker (δk). In addition for every worker, there are n new workers entering the economy, each of whom require k units of capital (nk) to keep capital per worker unchanged.
3. The basic intuition is that unless you replace worn out capital and provide each new worker with the same amount of capital, then capital per worker in the economy will fall. So the level of break-even investment per worker (the amount of investment per worker necessary to leave capital per worker unchanged) is equal to $(n + \delta)k$

III. THE SOLOW DIAGRAM

- The two equations of the Solow Model can be written in per capita terms as

$$\begin{aligned} y &= k^\alpha \\ \dot{k} &= sy - (n + \delta)k \end{aligned}$$

- We combine these 2 equations graphically into a diagram, known as a Solow diagram. The Solow diagram plots savings per worker and the break-even level of investment per worker as functions of the capital stock per worker (k).
- Some features of the diagram:
  1. $(n + \delta)k$ is a straight line from the origin with slope = $(n + \delta)$.
  2. $y = k^\alpha$ is a concave function. For those of you who are mathematically inclined: you can show that $\frac{dy}{dk} = \alpha k^{\alpha-1} > 0$ is an increasing function of k and that the slope of y falls as k increases: $\frac{d^2y}{dk^2} = \alpha(\alpha - 1)k^{\alpha-2} < 0$ because $\alpha < 1$.
  3. sy is just a dampened version of the y curve because s is a constant fraction between 0 and 1.
- The Solow diagram looks as follows.



III. THE STEADY STATE

- From this diagram, we can show that the economy has a steady state level of capital per worker, which we denote as $k*$, that it gravitates towards over time.

  1. When $k < k^*$, savings per worker > break-even investment per worker, i.e. $sy > (n+\delta)k$: k increases over time, i.e. $\dot{k} > 0$.
  2. When $k > k^*$, savings per worker < break-even investment per worker, i.e. $sy < (n+\delta)k$,: k decreases over time, i.e. $\dot{k} < 0$.
  3. At $k = k*$, savings per worker = break-even investment per worker, i.e. $sy = (n + \delta)k$: k is constant over time, i.e. $\dot{k} = 0$

- Once the economy reaches k^*, the steady state, it stays there. The intuition is that the amount of saving/investing the economy does at that point is exactly enough to cover the break even needs so the number of machines per worker stays unchanged. [By the way there is another (not quite as interesting) steady state in the model. Can you find it?]

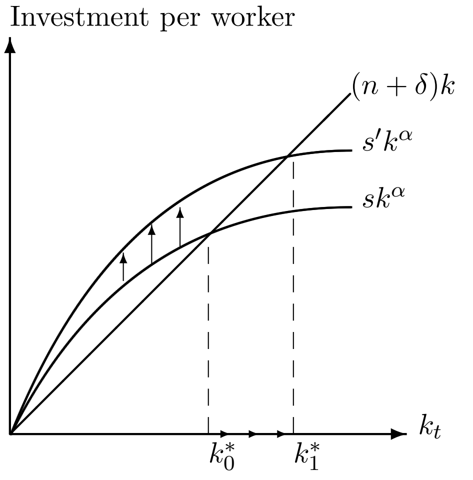
Changes in the Steady State

- Now that we have identified the steady state of the economy and figured out how to draw the Solow diagram, we are ready to analyze the impact of particular changes to the economy.
- These changes fall into two categories:

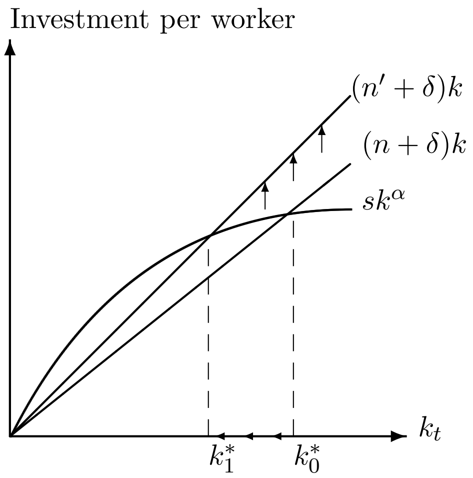
  1. Changes in the economy that cause the savings line or the break-even line to shift.
  2. Changes in the economy that cause a move along the savings line or the break-even line.

- The first category consists of changes that change the shape of the savings line or the break even line. Given that the savings line is sk^{α} and the break-even line is $(n + \delta)k$, we can see that changes in s, α, n or δ will cause shifts in the curves - the first two moving the savings line and the last two moving the break even line.
- The second category consists of things that suddenly change the level of k (capital per worker) in the economy. Since k is on the horizontal axis of the Solow diagram, a sudden change in k will mean a sudden movement along both the savings line and the break even line. Sudden changes in k can be driven by sudden changes in K or sudden changes in L.

- EXAMPLE 1: An increase in the savings rate. An increase in s will shift the savings line up. At the initial steady state, we will be investing more than we need to break even so the level of capital per worker will rise. This will continue until the new steady state at k_1^* is reached.



- EXAMPLE 2: An increase in the growth rate of population. An increase in n will shift the break-even investment line up. At the initial steady state, we will be investing less than we need to break even so the level of capital per worker will fall. This will continue until the new steady state at k_1^* is reached.



- EXAMPLE 3: A sudden increase in the capital stock. If K rises suddenly then k will rise as well, say to k_1 on the Solow diagram. At k_1, we will be investing less than we need to break even so the level of capital per worker will fall. This will continue until the economy returns to the old steady state.