Fall Semester '05-'06
Akila Weerapana

Lecture 9: The Solow Model With Technology

I. OVERVIEW

- In the last lecture we examined the behavior of the Solow model away from the steady state. We found that the predictions of the Solow model would enable us to have intuitive explanations for differences in growth rates among countries that are away from steady state.
- We also showed that the Solow model is unable to explain differences in steady state growth rates across countries: the model predicts that all countries have a per-capita output growth rate of zero.
- Furthermore, there is no possibility for sustained economic growth since all economies converge to a long run growth rate of zero.
- Today's lecture extends the Solow model in order to incorporate technology. Hopefully, adding technology to the model will preserve the sensible predictions of the basic Solow model and improve some of the less-sensible predictions of the basic model.
- The structure of the lectures is the same: we will first derive the model with technology algebraically. Then we will use a modified Solow diagram to perform comparative static analysis and conclude by assessing the predictions of the model.

II. THE SOLOW MODEL WITH TECHNOLOGY

- What is technology? Essentially, technology is ideas or knowledge. In particular, technology is knowledge about how to put inputs together to make more output. Essentially, having better technology means being able to produce more output from a given quantity of inputs.
- What we classify as "technology" can range from engineering knowledge to business innovations, like assembly line production, to service concepts like Wal-Mart or multiplex movie theaters.

The Production Function

- The production function in the Solow model is modified to include technology, and can be written as

$$Y = K^{\alpha}(AL)^{1-\alpha}$$

- The particular technology described in the above equation is known as **labor-augmenting technology**, basically ideas that help make labor more productive. We can also specify technology as being of the **capital augmenting** type, i.e. ideas that make capital more productive or of the **total factor productivity** type, ideas that make both capital and labor more productive.

- It turns out that once we work with the model containing labor-augmenting technology, the other two types are easy to work with.
- AL is known as the number of effective units of labor. Technology makes workers more effective; each worker counts as A workers in production so there are 'effectively' AL workers in the economy.
- We assume that the growth rate of technology is exogenous $\frac{\dot{A}}{A} = g$. In other words our model doesn't explicitly determine what causes technology to grow; instead we assume the growth rate of technology is equal to a constant rate, g.
- As before we will assume that the growth rate of the labor force is given exogenously (from outside the model) equal to $\frac{\dot{L}}{L} = n$.
- By taking logs and differentiating the production function we can obtain the following:

$$\begin{aligned} Y &= K^{\alpha}(AL)^{1-\alpha} \Rightarrow \ln(Y) = \alpha \ln(K) + (1-\alpha)[\ln(L) + \ln(A)] \\ \Rightarrow \frac{\dot{Y}}{Y} &= \alpha \frac{\dot{K}}{K} + (1-\alpha)[\frac{\dot{L}}{L} + \frac{\dot{A}}{A}] \\ &= \alpha \frac{\dot{K}}{K} + (1-\alpha)(n+g) \end{aligned}$$

- From this we can see that in order to understand the growth rate of Y we need to understand the growth rate of K, which is determined endogenously (within the model).
- This leads us to the second equation in the Solow model: the capital accumulation equation which describes how the capital stock evolves over time.

The Capital Accumulation Equation

- The second equation of the model is exactly the same

$$\dot{K} = sY - \delta K$$

- The major difference from the basic Solow model is the transformation of the model. Instead of working with per-worker variables we instead work with per-effective worker variables. We define $\tilde{k} = \frac{K}{AL}$ and $\tilde{y} = \frac{Y}{AL}$
- $\tilde{k}$ is known as **capital per effective worker**, and $\tilde{y}$ is known as **output per effective worker**. This follows from the concept that, since each worker is worth A workers, there are effectively AL workers in the economy.
- We chose to normalize this way because we need to take into account both the presence of labor and technology in the model. As you will see below rewriting the model as a function of a single variables requires us to move to per-effective worker instead of per-worker terms.
- Note the relationship between the per-worker variables and the per-effective worker variables.

$$\begin{aligned} \tilde{k} &\equiv \frac{K}{AL} \equiv \frac{K/L}{A} = \frac{k}{A} \\ \tilde{y} &\equiv \frac{Y}{AL} \equiv \frac{Y/L}{A} = \frac{y}{A} \end{aligned}$$

- So, if the idea of 'effective workers' is unintuitive to you, another way to think of is as 'capital per worker per unit of technology!'
- The production function can be written in per-effective worker terms as $\frac{Y}{AL} = \frac{K^\alpha (AL)^{1-\alpha}}{AL} \equiv \frac{K^\alpha}{(AL)^\alpha}$ which means that

$$\tilde{y} = \tilde{k}^\alpha$$

- The capital accumulation equation can be written in per-effective worker terms as well.

$$\begin{aligned}
\tilde{k} &= \frac{K}{AL} \Rightarrow \ln(\tilde{k}) = \ln(K) - \ln(AL) \equiv \ln(K) - \ln(A) - ln(L) \\
\Rightarrow \frac{\dot{k}}{k} &= \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} \\
&= \frac{(sY - \delta K)}{K} - g - n \equiv \frac{(sY)}{K} - \delta - g - n \\
&= \frac{(sY/(AL))}{K/(AL)} - (\delta + g + n) \equiv \frac{(s\tilde{y})}{\tilde{k}} - (\delta + g + n) \\
\Rightarrow \dot{\tilde{k}} &= s\tilde{y} - (\delta + g + n)\tilde{k}
\end{aligned}$$

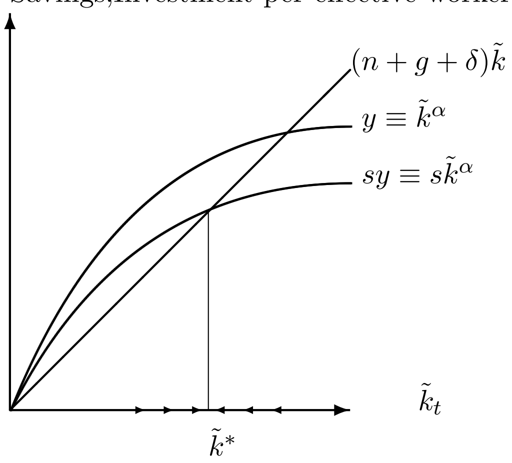
- This equation states that net investment per effective worker (the change in the capital stock per effective worker) is the difference between gross investment per effective worker ($s\tilde{y}$) and the level of break-even investment per worker $(n + g + \delta)\tilde{k}$.
- The intuition underlying this equation is not too difficult to follow:
  1. Gross investment per effective worker is equal to saving per effective worker (follows from the national income accounting identity gross investment = savings).
  2. Some of this new capital per effective worker must be used to replenish depreciated capital per worker ($\delta\tilde{k}$). In addition for every worker, there are $n + g$ new effective workers (labor growing at n and technology growing at g) entering the economy, each of whom require $\tilde{k}$ units of capital $(n+g)\tilde{k}$) to keep capital per effective worker unchanged.
  3. The basic intuition is that unless you replace worn out capital, provide each new worker with the same amount of capital, and invest in capital to keep up with technology then capital per effective worker in the economy will fall. So the level of break-even investment per effective worker (the amount of investment per effective worker necessary to leave capital per effective worker unchanged) is equal to $(n + g + \delta)\tilde{k}$

III. THE SOLOW DIAGRAM

- The two equations of the Solow Model can be written in per effective worker terms as

$$\begin{aligned}
\tilde{y} &= \tilde{k}^\alpha \\
\dot{\tilde{k}} &= s\tilde{y} - (n + g + \delta)\tilde{k}
\end{aligned}$$

- We combine these 2 equations graphically into a new Solow diagram. The new Solow diagram plots savings per effective worker and the break-even level of investment per effective worker as functions of the capital stock per effective worker ($\tilde{k}$).
- Some features of the diagram:
  1. $(n+g+\delta)\tilde{k}$ is a straight line from the origin with slope $= (n+g+\delta)$.
  2. $\tilde{y} = \tilde{k}^\alpha$ is a concave function. For those of you who are mathematically inclined: you can show that $\frac{d\tilde{y}}{d\tilde{k}} = \alpha\tilde{k}^{\alpha-1} > 0$ is an increasing function of k and that the slope of $\tilde{y}$ falls as $\tilde{k}$ increases: $\frac{d^2\tilde{y}}{d\tilde{k}^2} = \alpha(\alpha-1)\tilde{k}^{\alpha-2} < 0$ because $\alpha < 1$.
  3. $s\tilde{y}$ is just a dampened version of the $\tilde{y}$ curve because s is a constant fraction between 0 and 1.
- The Solow diagram looks as follows.



The Steady State

- From this diagram, we can show that the economy has a steady state level of capital per effective worker, which we denote as $\tilde{k}^*$, that it gravitates towards over time.
  1. When $\tilde{k} < \tilde{k}^*$, savings per effective worker > break-even investment per effective worker, i.e. $s\tilde{y} > (n+g+\delta)\tilde{k}$: $\tilde{k}$ increases over time, i.e. $\dot{\tilde{k}} > 0$.
  2. When $\tilde{k} > \tilde{k}^*$, savings per effective worker < break-even investment per effective worker, i.e. $s\tilde{y} < (n+g+\delta)\tilde{k}$: $\tilde{k}$ decreases over time, i.e. $\dot{\tilde{k}} < 0$.
  3. At $\tilde{k} = \tilde{k}^*$, savings per effective worker = break-even investment per effective worker, i.e. $s\tilde{y} = (n+g+\delta)\tilde{k}$: $\tilde{k}$ is constant over time, i.e. $\dot{\tilde{k}} = 0$

Describing the Economy at the Steady State

- At the steady state, we know that $\tilde{k}$ is constant ($\tilde{k} = \tilde{k}^*$) Therefore $\frac{\dot{\tilde{k}}}{\tilde{k}} = 0$.

- Since $\tilde{y} = \tilde{k}^{\alpha}$, we can first take logs to get $\ln \tilde{y} = \alpha \ln \tilde{k}$, and then take derivatives of both sides to get $\frac{\dot{\tilde{y}}}{\tilde{y}} \equiv \alpha \frac{\dot{\tilde{k}}}{\tilde{k}} = 0$.

- We also know that $\tilde{k} = \frac{K}{AL} = \frac{k}{A}$. Taking logs and differentiating both sides we get $\frac{\dot{\tilde{k}}}{\tilde{k}} \equiv \frac{\dot{k}}{k} - \frac{\dot{A}}{A} \Rightarrow \frac{\dot{k}}{k} = \frac{\dot{\tilde{k}}}{\tilde{k}} + \frac{\dot{A}}{A}$

- Since $\tilde{k}$ is constant at steady state we get that $\frac{\dot{k}}{k} = \frac{\dot{A}}{A} = g$

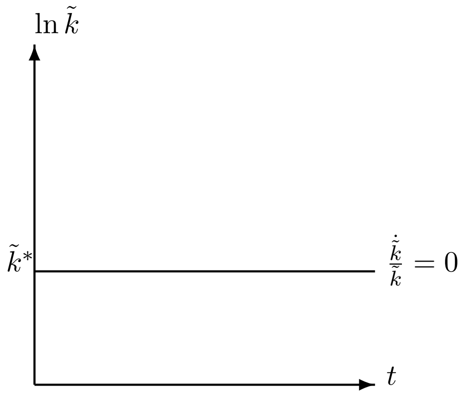
- We can use the fact that $\tilde{y} = \frac{y}{A}$ to show that $\frac{\dot{y}}{y} = \frac{\dot{\tilde{y}}}{\tilde{y}} + \frac{\dot{A}}{A}$ and therefore show that $\frac{\dot{y}}{y} = \frac{\dot{A}}{A} = g$

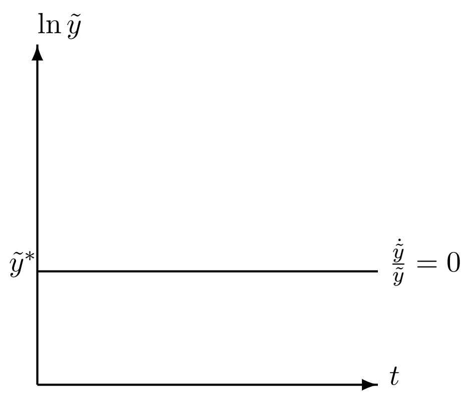
- Finally, since $k = \frac{K}{L}$, taking logs and differentiating both sides we get $\frac{\dot{k}}{k} \equiv \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \Rightarrow \frac{\dot{K}}{K} = \frac{\dot{k}}{k} + \frac{\dot{L}}{L}$

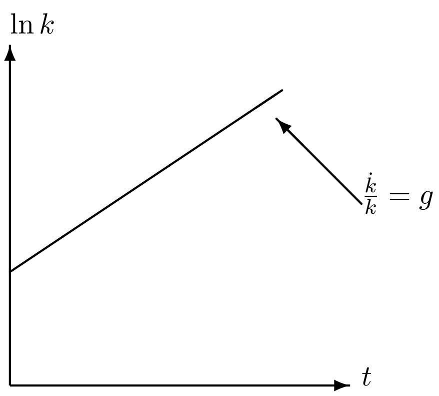
- Since k is growing at a rate g in steady state we get that $\frac{\dot{K}}{K} = g + n$

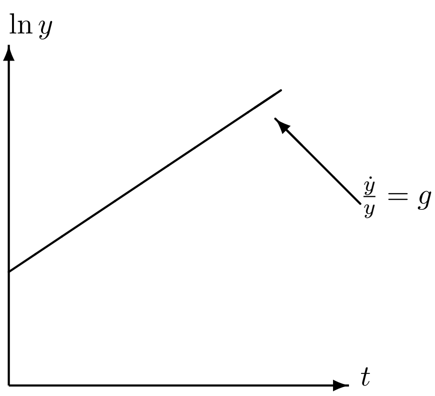
- Similarly, we can use the fact that $y = \frac{Y}{L}$ to show that $\frac{\dot{Y}}{Y} = \frac{\dot{y}}{y} + \frac{\dot{L}}{L}$ and therefore show that $\frac{\dot{Y}}{Y} = g + n$

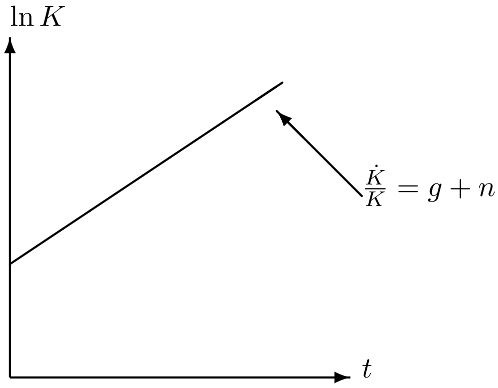
- We can also graph the paths of $\tilde{k}$, $\tilde{y}$, k, y, K and Y over time. The steady state time paths are:

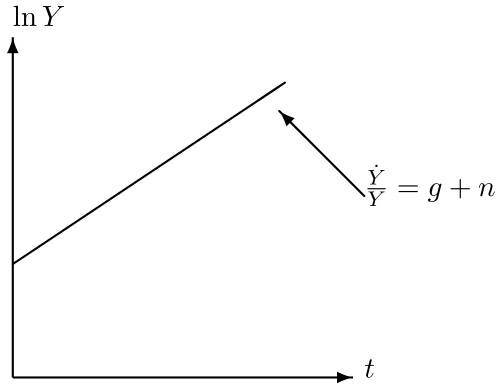









$\ln K$
$\frac{\dot{K}}{K} = g + n$
t


$\ln Y$
$\frac{\dot{Y}}{Y} = g + n$
t