Fall Semester '07-'08
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LECTURE 4: DIFFERENTIAL CALCULUS

I. INTRODUCTION

- In the last couple of classes, we talked about some applications of the exponential and logarithmic functions in Economics. While the theory was mostly review, some of the applications may have been new to you. Today's lecture will also follow a similar pattern: we will review some basic concepts in differential calculus and then spend the majority of the time discussing economic applications of these concepts.

- As we discussed in the first lecture, solving an economic model usually entails deriving a system of equations that describe endogenous variables as functions of exogenous variables. We can then use differential calculus to do "comparative statics analysis": look at how the endogenous variables change in response to changes in the exogenous variables. This is an abstract way of analyzing the impact of policy changes on the economy without performing experiments.

- If you need to brush up on your calculus, I urge you to do so immediately. Chapters 6 and 7 in Klein will be a good starting point. Chapter 8 in Klein will be the material more immediately relevant to this lecture.

II. THEORY

Differentials

- In basic calculus, you learned that the **derivative**, with respect to x, of the function $y = f(x)$ was defined as
$$\left(\frac{dy}{dx}\right) = f'(x) = \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right]$$

- The derivative $f'(x)$ describes the slope of the function $f(x)$.

- In a later class, you would have learned about the **differential** dy which is described by the equation
$$dy = f'(x)dx$$

- If dx denotes a given change in x, then this equation states that the differerential dy is equal to the slope of the line times the change in x. Geometrically, as Figure 6.9 in Klein shows, the differential dy is an approximation to the actual change in y as a result of the change in x given by dx. The smaller dx is, the better approximation dy becomes. Klein also points out that the close the function is to being linear, the better approximation dy is to the actual change in y.

- If the geometry is puzzling, use your calculus knowledge to develop intuition about the differential. First, dy and dx represent small changes in the respective variables. Second, the derivative $f'(x)$ measures how f(x) changes when there is a infinitesimally small unit change in x. The differential formula states that the change in y (dy) can be found by multiplying how much y changes per-unit change in the x variable ($f'(x)$) by the number of units that x actually changed (dx).

- So as long as you know how to take a derivative, you can calculate the differential dy for any univariate function of the form y=f(x). If you are not yet comfortable, read section 6.4 in Klein and do the exercises at the end of the section

Total Differentials

- The next task is to look at how to calculate the differential for a multivariate function $y = f(x_1, x_2, ..., x_n)$. The basic idea is similar: changes in y are brought about through changes in any of the x_i variables, with the partial derivative $\frac{\partial y}{\partial x_i}$ describing the per-unit impact of these changes on y

- If we use the short hand notation $f_i = \frac{\partial y}{\partial x_i}$, the **total differential** of the function dy can be written as $dy = f_1 dx_1 + f_2 dx_2 + ...f_n dx_n$

- Intuitively, the change in y can be understood as being brought about by changes in any of the exogenous variables. The magnitude of this change in y can be found by multiplying the partial derivative $\frac{\partial y}{\partial x_i}$, which measures the impact on y of an infinitesimally small change in x_i, by the actual change dx_i.

  Example

- Let's do a short mathematical example before proceeding to the wide array of economic applications that the total differential can be used for. Consider the function $z = 5x^2 + 6xy + 3y^2$. We can calculate the partial derivatives $\frac{\partial z}{\partial x} = 10x + 6y$ and $\frac{\partial z}{\partial y} = 6x + 6y$, and then write the total differential as $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (10x + 6y)dx + (6x + 6y)dy$

- If you need to do more problems to feel comfortable, you can try the problems at the end of section 8.4 in Klein.

  Implicit Function Theorem

- Most of the functions that we typically discuss have the explicit form $y = f(x_1, x_2, ..., x_n)$, i.e. the dependent variable can be written explicitly as a function of the independent variables. With this formulation, taking the partial derivative of y with respect to any of the x variables is fairly straightforward.

- Sometimes, we have to deal with functions in which it is not always possible to solve for y as an explicit function of the x variables. In that case what we have is an **implicit function** of the form $F(y, x_1, x_2, ..., x_n) = 0$.

- We can treat this function as being a special case of an explicit function of the form $z = F(y, x_1, x_2, ...x_n)$ where z is always zero. By taking the total differential of the equation we get $dz = F_y dy + F_{x_1} dx_1 + F_{x_2} dx_2 + ... + F_{x_n} dx_n$. Since z is a constant, we know that dz=0 always. So we get $0 = F_y dy + F_{x_1} dx_1 + F_{x_2} dx_2 + ... + F_{x_n} dx_n$. We can then calculate the impact of a change in a given variable x_i, holding all other variables constant as $0 = F_y dy + F_{x_i} dx_i$ for $i = 1...n$, which can be re-written as

$$\frac{dy}{dx_i} = \frac{-F_{x_i}}{F_y}$$

- This is an important result that shows how to calculate the impact of a change in the independent variable on the dependent variable even in cases where we only have an implicit relationship between the two. A mathematical example is given below; economic applications will be explored later on in the lecture.

Example

- Consider calculating $\left(\frac{dy}{dx}\right)$ using the function $y^3 - 2y^2x + 6yx^2 - 30 = 0$. This is an implicit function of y and x of the form F(y,x)=0, it would be extremely tedious to write y as an explicit function of x and try to rewrite it in the form y=f(x).
- Using the implicit function theorem, $\left(\frac{dy}{dx}\right) = \left(\frac{-F_x}{F_y}\right)$ which simplifies to

$$\left(\frac{dy}{dx}\right) = \frac{-(-2y^2 + 12xy)}{3y^2 - 4yx + 6x^2} = \frac{(2y^2 - 12xy)}{3y^2 - 4yx + 6x^2}$$

Chain Rule

- Often we run across functions in which the arguments of the function are themselves functions of a different variable. Consider a function of the form $y = f(x, w)$ where $x = g(w)$. Calculating the total differential is tricky: we have to be aware that changes in w can affect y in a couple of ways.
- The total differential for this function can be written as $dy = f_x dx + f_w dw$. Since x is a function of w, the differential of x can be written as $dx = g'(w)dw$. Combining these two we get $dy = f_x g'(w)dw + f_w dw$ This technique is called the **chain rule**.
- Intuitively, the change in y is brought about by changes in w working through two channels. First, changes in w directly affect y. The magnitude is given by the change in w multiplied by the partial derivative of f with respect to w, f_w, which measures the impact of a small change in w on y, holding x constant.
- However, when w changes, x does not stay constant. A small change in w brings about a change in x equivalent to $g'(w)dw$, i.e. the change in w multiplied by the derivative of x with respect to w. This change is multiplied by the partial derivative of f to x, f_x, which measures the impact of a small change in x on y, all else constant.

Example

- Think about the consumption function $c = C(Y, \tau)$ where $\tau = T(Y)$, i.e. consumption is assumed to be a function of income and of taxes paid by the individual, but taxes themselves are related to income. When we examine the impact of changes in income on consumption we calculate the total differential: $dc = C_Y dY + C_\tau d\tau$. But we have to keep in mind that $d\tau = T'(Y)dY$. Combining the two, we get $dc = C_Y dY + C_\tau T'(Y)dY$, which states that the change in Y has both a direct effect and an indirect effect, working through income tax changes, on consumption.

III. APPLICATIONS

Elasticity

- One of the most common applications of differentials is in calculating elasticities. The **elasticity** of a variable y with respect to a variable x, denoted $\epsilon_{y,x}$ describes the percentage change in y associated with a 1% change in x. The formula for calculating point elasticities can be expressed in terms of differentials as

$$\epsilon_{y,x} = \frac{\left(\frac{dy}{y}\right)}{\left(\frac{dx}{x}\right)}$$

- In other words the numerator gives us the percentage change in y (the change in y divided by the level of y) while the denominator gives us the percentage change in x (the change in x divided by the level of x).
- Economist often use another, equivalent formula for calculating elasticities, namely

$$\epsilon_{y,x} = \frac{d(lny)}{d(lnx)}$$

- The two formulas are equivalent by considering that $z = ln(y) \Rightarrow dz = \frac{1}{y}dy \Rightarrow d(lny) = \frac{dy}{y}$ and $d(lnx) = \frac{dx}{x}$

Example

- Calculate the price elasticity associated with a demand function of the form $Q = aP^{1/3}$. The differential is $dQ = \frac{1}{3}aP^{(-2/3)}dP$. Dividing both sides by Q we get $\frac{dQ}{Q} = \frac{\frac{1}{3}aP^{(-2/3)}}{Q}dP = \frac{1}{3}\frac{dP}{P}$. So $\epsilon_{Q,P} = \frac{\left(\frac{dQ}{Q}\right)}{\left(\frac{dP}{P}\right)} = \frac{1}{3}$.
- We could have also calculated this as $Q = aP^{1/3} \Rightarrow \ln Q = \ln a + \frac{1}{3}\ln P$. Taking the differential we get $d(\ln Q) = \frac{1}{3}d(\ln P)$, which again gives us $\epsilon_{Q,P} = \frac{d(lnQ)}{d(lnP)} = \frac{1}{3}$

Instantaneous Growth Rates

- The second application of a differential is to use it to derive the **instantaneous growth rate** of a variable. In general, we know that the growth rate of a variable was defined as the % rate of change of that variable, i.e. the % change in the variable divided by the length of the period.

- If X is a variable that is growing continuously, the instantaneous growth rate of that variable is described by the formula
$$g_y = \frac{dy/y}{dt}$$
- Since the numerator tells us the % change in y and dt tells us the elapsed time period, the ratio describes the instantaneous % rate of change in the variable, i.e. its instantaneous growth rate.
- As with elasticities, economists use another formula involving natural logarithms to describe the instantaneous growth rate of a variable. That formula is $g_y = \frac{d(lny)}{dt}$. Once again this is equivalent to the above because $d(lny) = dy/y$

  Example

- If the capital stock is growing continuously at a rate m, we can describe its value at time t by the following equation $K_t = K_0 e^{mt}$. The differential of K gives us $dK_t = mK_0 e^{mt} dt = mK_t dt$. Rearranging, we get $\frac{dK_t}{K_t} = mdt \Rightarrow \frac{\frac{dK_t}{K_t}}{dt} = m$. So $g_{K_t} = m$
- Alternatively, we could have written $\ln K_t = \ln K_0 + mt$. The differential is $d(\ln K_t) = mdt$. So $g_{K_t} = \frac{d(\ln K_t)}{dt} = m$

Growth Rates of Products and Ratios

- We can also derive some general rules for calculating the growth rates of variables that are functions of other variables whose growth rates are known. Let A and B be variables that have instantaneous growth rates g_A and g_B respectively.
- Suppose C=AB. We can use a technique that one of my old professors termed the TLAD (Take Logs and Differentials) technique) to calculate the instantaneous growth rate of C. Taking logs $\ln C = \ln A + \ln B$ The differential gives us $d(lnC) = d(\ln A) + d(\ln B)$. Dividing by dt we get $\frac{d(lnC)}{dt} = \frac{d(lnA)}{dt} + \frac{d(lnB)}{dt}$ which implies that $g_C = g_A + g_B$
- Suppose C=A/B. Taking logs $\ln C = \ln A - \ln B$ The differential gives us $d(lnC) = d(\ln A) - d(\ln B)$. Dividing by dt we get $\frac{d(lnC)}{dt} = \frac{d(lnA)}{dt} - \frac{d(lnB)}{dt}$ which implies that $g_C = g_A - g_B$
- Suppose $C = \alpha A$, where α is a constant. Taking logs $\ln C = \ln \alpha + \ln A$ The differential gives us $d(lnC) = d(\ln A)$. Dividing by dt we get $\frac{d(lnC)}{dt} = \frac{d(lnA)}{dt}$ which implies that $g_C = g_A$
- Suppose $C = A^{\alpha}$. Taking logs, $\ln C = \alpha \ln A$ The differential gives us $d(lnC) = \alpha d(\ln A)$. Dividing by dt we get $\frac{d(lnC)}{dt} = \alpha \frac{d(lnA)}{dt}$ which implies that $g_C = \alpha g_A$

  Example 1

- Suppose GDP (Y) is growing continuously at a rate g and that population (N) is growing at a rate n, What is the growth rate of GDP per capita (y=Y/N). Since $y = \frac{Y}{N}$, taking logs on both sides we get $\ln y = \ln Y - \ln N$. Calculating the differential, and dividing by the change in time gives us $\frac{d(\ln y)}{dt} = \frac{d(\ln Y)}{dt} - \frac{d(\ln N)}{dt}$, which implies that $g_y = g_Y - g_N$

Example 2

- Suppose your utility function is of the form $U(C) = \sqrt{C}$. If consumption is growing at a rate n, what is the growth rate in the utility you derive from that consumption? Taking logs we get $\ln U = 1/2ln(C)$. From this we can calculate the differential $d\ln U = 1/2d\ln C$, dividing by the change in time we get $g_U = 1/2g_C$.

Growth Accounting

- One of the key contributions to economics of Robert Solow was the procedure known as **growth accounting**, which was a technique that helped decompose economic growth into its contributing factors
- Suppose you have a generic production function $Y = F(A, K, L)$ where K is capital, L is labor and A is technology. We can write the differential of Y as

$$dY = F_K dK + F_L dL + F_A dA$$

- Dividing through by dt we get

$$\frac{dY}{dt} = F_K \frac{dK}{dt} + F_L \frac{dL}{dt} + F_A \frac{dA}{dt}$$

- The growth rate of Y can then be expressed as

$$\begin{aligned}
\frac{\frac{dY}{dt}}{Y} &= F_K \frac{\frac{dK}{dt}}{Y} + F_L \frac{\frac{dL}{dt}}{Y} + F_A \frac{\frac{dA}{dt}}{Y} \\
g_Y &= \frac{KF_K}{Y} \frac{\frac{dK}{dt}}{K} + \frac{LF_L}{Y} \frac{\frac{dL}{dt}}{L} + \frac{AF_A}{Y} \frac{\frac{dA}{dt}}{A} \\
g_Y &= \left(\frac{KF_K}{Y}\right) g_K + \left(\frac{LF_L}{Y}\right) g_L + \frac{AF_A}{Y} g_A
\end{aligned}$$

- This can be written in short-hand as

$$g_Y = s_K g_K + s_L g_L + s_A g_A$$

where s_K is the share of output produced by capital, s_L is the share of output produced by labor, and s_A is the share of output produced by technology.

Simple Comparative Static Exercises

- We now have the tools to do some simple **comparative statics** exercises where we look at the impact of changes in exogenous variables on endogenous variables.

Supply and Demand

- Consider the solution to the simple supply demand model $D(Y,P) = S(W,P)$. Taking the total differential gives us $D_Y dY + D_P dP = S_W dW + S_P dP$ or equivalently that

$$D_Y dY - S_W dW = (S_P - D_P) dP$$

- We can now look at the impact of a change in income (Y) on the equilibrium price (P), holding all else constant. This can be done by setting $dW = 0$ in the above equation and solving for $\frac{dP}{dY} = \frac{D_Y}{(S_P - D_P)}$. We can then use economic knowledge to interpret and sign this derivative.
- Since quantity supplied increases with price while quantity demanded falls with price we know that $S_P > 0$ and $D_P < 0$. Therefore, the denominator is going to be positive. So the sign of $\frac{dP}{dY}$ will be the same as the sign of D_Y .
- If the good is normal, i.e. $D_Y > 0$, then higher income leads to a greater quantity demanded and an increase in price, i.e. $\frac{dP}{dY} > 0$. If the good is inferior, i.e. $D_Y < 0$, then higher income leads to less quantity demanded and a decrease in price, i.e. $\frac{dP}{dY} < 0$
- Similarly, we can look at the impact of a change in input prices (W) on the equilibrium price (P), holding all else constant. This can be done by setting $dY = 0$ in the above equation and solving for $\frac{dP}{dW} = \frac{-S_W}{(S_P - D_P)}$.
- Since quantity supplied decreases with input prices, $S_W < 0$ and thus the numerator is positive. Since we have shown that the denominator is going to be positive. So the sign of $\frac{dP}{dW}$ is > 0, i.e. output prices rise with input prices.

Saving and Investment

- Consider the following simple macroeconomic model

$$\begin{aligned} Y &= C + I + \bar{G} \\ S &= Y - C - \bar{T} \\ I &= I(r) \\ C &= C(Y, r) \\ \bar{D} &= \bar{G} - \bar{T} \end{aligned}$$

where S is private saving, G is government purchases, T is taxes and D is the budget deficit. A vertical bar indicates an exogenous variable.

- We are given the following pieces of information $0 < C_Y < 1, C_r \leq 0, I_r < 0$.
- The saving function can be written as $S = S(Y,r) = Y - C(Y,r) - T$ with $S_Y > 0$ and $S_r > 0$ given our assumptions above.
- We can therefore express the solution to this model as

$$S(Y,r) = I(r) + D$$

- Taking the total differential gives us

$$S_Y dY + S_r dr = I'(r)dr + dD$$

- We can now look at the impact of a change in the budget deficit (D) on the equilibrium interest rate (r), holding all else constant (note that we are treating Y as an exogenous variable here). This can be done by setting $dY = 0$ in the above equation and solving for $\frac{dr}{dD} = \frac{1}{(S_r - I'(r))}$. We can then sign this derivative as being > 0, given the assumptions made in setting up the model.

- We can also look at the impact of a change in GDP (Y) on the equilibrium interest rate (r), holding all else constant. This can be done by setting $dD = 0$ in the above equation and solving for $\frac{dr}{dY} = \frac{S_Y}{(I'(r) - S(r))}$. We can then sign this derivative as being < 0, given the assumptions made in setting up the model.