Fall Semester '07-'08
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LECTURE 7: SOLVING LINEAR SYSTEMS

I. INTRODUCTION

- In the last lecture, we laid out the basics of linear algebra: we defined what matrices are, described elementary operations such as transposition, addition and multiplication of matrices.
- We also defined a number that is unique to square matrices, called the determinant. We showed simple techniques for calculating determinants of small matrices (upto 3×3) and discussed a more complicated method using cofactors for calculating the determinant of an $n \times n$ matrix.
- We also defined the inverse of a matrix. We showed that only square matrices are invertible and that not all square matrices are invertible either. The determinant of a matrix tells us whether or not it is invertible. If the determinant is zero, the matrix is not invertible (and vice versa).
- Today's lecture discusses how to represent and solve systems of linear equations. One way to solve such a system is to calculate the inverse of a matrix and use the inverse to solve the system of linear equations. Given the complications associated with solving for the inverse of a matrix approach, we will discuss an alternative method to solve the system of equations.

II. REPRESENTING LINEAR SYSTEMS IN MATRIX NOTATION

Linear Equations

- Basic linear algebra is widely used to describe a system of linear equations. Consider the following simple 3 equation system.

$$\begin{aligned} 3x + 6y - 5z &= 15 \\ -x + y + 11z &= 1 \\ 2x + y + x &= 4 \end{aligned}$$

- We can represent this in matrix notation as

$$\begin{bmatrix} 3 & 6 & -5 \\ -1 & 1 & 11 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 1 \\ 4 \end{bmatrix}$$

Linear Economics Models

- We can do the same for simple linear algebraic economic models as well. Consider a simple supply demand model described by the following equations

$$\begin{aligned} Q^d &= aY - bP \\ Q^s &= dP - cW \\ Q^d &= Q^s \end{aligned}$$

- We will first re-write the model with all the endogenous variables on the left hand side (LHS).

$$\begin{aligned} Q^d + bP &= aY \\ Q^s - dP &= -cW \\ Q^d - Q^s &= 0 \end{aligned}$$

- We can represent this model as a system of equations that relate the endogenous variables (P, Q^s and Q^d) to the exogenous variables (Y and W) as follows.

$$\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} Q^d \\ Q^s \\ P \end{bmatrix} = \begin{bmatrix} aY \\ -cW \\ 0 \end{bmatrix}$$

- Note that we have not discussed how to solve this system yet, the first thing to do is to be comfortable describing systems of equations using matrix algebra. Read through Chapter 4.1 to 4.3 of Klein to make sure you are comfortable with matrix notation.

III. SOLVING LINEAR SYSTEMS

- The above system can be thought of as a system of equations in the form $AY = BX$ where A and B are matrices of constants, Y is a vector of endogenous variables and X is a vector of exogenous variables.

- Sometimes, we write this as $AY = b$ where b is a column vector whose elements are functions of exogenous variables.

- For instance, the supply/demand model can be written in the form $AY = BX$ where

$$A = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{bmatrix}, Y = \begin{bmatrix} Q^d \\ Q^s \\ P \end{bmatrix}, B = \begin{bmatrix} a & 0 \\ 0 & -c \\ 0 & 0 \end{bmatrix}, X = \begin{bmatrix} Y \\ W \end{bmatrix}$$

- Alternatively, we can write it in the form $AY = b$ where

$$A = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{bmatrix}, Y = \begin{bmatrix} Q^d \\ Q^s \\ P \end{bmatrix}, b = \begin{bmatrix} aY \\ -cW \\ 0 \end{bmatrix}$$

- If the matrix A is invertible, then we can calculate the solution to a system of linear equations of the form $AY = b$ as $Y = A^{-1}b$.

Existence of a Solution

- A is only invertible if it has a non-zero determinant, i.e. A is a non-singular matrix. Under what conditions does A have a non-zero determinant? Can we look at a system of equations that represents an economic model and be able to intuitively grasp that the model has a solution? These questions can be answered by using another property of a matrix, known as the **rank**.
- The rank of a square matrix (which is what we are concerned about here) is the number of linearly independent rows that exist in that matrix. What is a linearly independent row? A linearly independent row is one which does not equal a linear combination of any other rows in that matrix. A couple of examples will illustrate the point.
- Consider the matrices

$$A = \begin{bmatrix} 3 & 7 & 0 \\ 2 & 5 & 3 \\ 1 & 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 3 & 7 & 0 \\ 2 & 5 & 3 \\ 4 & 10 & 6 \end{bmatrix}, C = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 4 & 12 & 8 \end{bmatrix}, D = \begin{bmatrix} 3 & 7 & 0 \\ 2 & 5 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

- Matrix A has rank 2: the first row is linearly dependent: it is the sum of the 2nd and 3rd rows. Matrix B has rank 2 as well: the 3rd row is 2 * the second row. Matrix C has rank 1 the 2nd row is twice as large as the first row and the 3rd row is four times as large as the first row. Matrix D has rank 3: none of the rows are a linear combination.
- An n*n square matrix that has rank ¡ n has a determinant of zero. So matrixes with linearly dependent rows (these matrices are said not to have "full rank") have a zero determinant and are therefore not invertible. Square matrices that have no linearly dependent rows (full rank matrices) have non-zero determinants and are invertible.
- In any economic problem where we write the model as a system of linear equations, we can only calculate the solution if the equations are linearly independent. Intuitively, this makes sense. If we are trying to solve for n variables, we need n unique equations. If we have n equations but one of them is linearly dependent then that equation does not contain any useful information. Given that $x + y = 2$ knowing that $2x + 2y = 4$ does not help us any, we still have as much info as we had with the equation $x + y = 2$.

Solving Using The Inverse Matrix

- Let's take the simple 3 equation supply demand model which we wrote out as a system of linear equations in the form

$$\begin{aligned} Q^d + bP &= aY \\ Q^s - dP &= -cW \\ Q^d - Q^s &= 0 \end{aligned}$$

- This is a system of equations in the form AY=b

$$\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} Q^d \\ Q^s \\ P \end{bmatrix} = \begin{bmatrix} aY \\ -cW \\ 0 \end{bmatrix}$$

- If A is invertible we can solve for the endogenous variables Y as $Y = A^{-1}b$. In order to find out whether A is invertible, we can calculate the determinant of A as $|A| = \begin{vmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ -1 & 1 & 0 \end{vmatrix} = b+d$

- So as long as b+d does not equal zero, A is invertible. Since the model was written in such a way that b and d are both positive parameters this condition will always hold.

- After some algebra, we can calculate the matrix of cofactors as $\begin{bmatrix} d & d & 1 \\ b & b & -1 \\ -b & d & 1 \end{bmatrix}$

- The inverse of A is then $A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{b+d} \begin{bmatrix} d & b & -b \\ d & b & d \\ 1 & -1 & 1 \end{bmatrix}$

- Since $Y = A^{-1}B$, we have

$$Y = \frac{1}{b+d} \begin{bmatrix} d & b & -b \\ d & b & d \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ -c \\ 0 \end{bmatrix} = \frac{1}{b+d} \begin{bmatrix} ad - bc \\ ad - bc \\ ac \end{bmatrix}$$

- So the solution to the system of linear equations is $Q^d = \frac{adY - bcW}{b+d}$, $Q^s = \frac{adY - bcW}{b+d}$ and $P = \frac{aY + cW}{b+d}$

- These results make intuitive sense. In equilibrium, quantity demanded and quantity supplied should be equal. Equilibrium price rises as income rises (demand goes up) and as input costs rise (supply goes down). An income increase should increase demand while an input price increase should decrease demand.

Solving using Cramer's Rule

- Hopefully, the last exercise firmly established in your mind that computers were invented purely to stop us from having to manually calculate the inverse of a matrix. Even a 3×3 matrix is fiendishly hard to invert manually without making careless algebraic errors.

- Thankfully, there is a technique called **Cramer's Rule** that enables us to solve a model of the form $AY = b$ without ever having to calculate an inverse.

- If there were any justice in the world, the person who invented this technique should receive great fame, fortune and the undying gratitude of every computerless student. Unfortunately, non of the books I referred to nor the web had any information about the man or woman, Cramer. What a cruel world we live in.

- Cramer's Rule is a devastatingly simple, yet brilliant, technique. Given a system of n equations of the form $AY = b$, where $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ we can use the following formula to solve for any y_i as $y_i = \frac{|A_i|}{|A|}$ where A_i is the matrix that we obtain from replacing the ith column of A with b!

- Essentially if we write $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ then

$$A_1 = \begin{bmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{bmatrix} \text{ etc .}$$

- With this technique solving the simple linear models described earlier becomes much easier.

- Given the system $\begin{bmatrix} 3 & 6 & -5 \\ -1 & 1 & 11 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 1 \\ 4 \end{bmatrix}$ we can calculate the solutions

$$x = \frac{1}{|A|} \begin{vmatrix} 15 & 6 & -5 \\ 1 & 1 & 11 \\ 4 & 1 & 1 \end{vmatrix} = \frac{123}{123}, y = \frac{1}{|A|} \begin{vmatrix} 3 & 15 & -5 \\ -1 & 1 & 11 \\ 2 & 1 & 1 \end{vmatrix} = \frac{246}{123}, z = \frac{1}{|A|} \begin{vmatrix} 3 & 6 & 15 \\ -1 & 1 & 1 \\ 2 & 1 & 4 \end{vmatrix} = \frac{0}{123}$$

- The solution $x = 1$, $y = 2$ and $z = 0$ is thus obtained much more easily with Cramer's Rule.

- The supply/demand model falls with much ease to the power of Cramer's Rule as well. Given $\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} Q^d \\ Q^s \\ P \end{bmatrix} = \begin{bmatrix} aY \\ -cW \\ 0 \end{bmatrix}$ we can solve for

$$Q^d = \frac{1}{|A|} \begin{vmatrix} aY & 0 & b \\ -cW & 1 & -d \\ 0 & 1 & 0 \end{vmatrix} = \frac{adY - bcW}{b+d} = Q^s, \text{ and } P = \frac{1}{|A|} \begin{vmatrix} 1 & 0 & aY \\ 0 & 1 & -cW \\ -1 & 1 & 0 \end{vmatrix} = \frac{aY + cW}{b+d}$$

IV. ECONOMIC APPLICATIONS

Tax Incidence

- One of the most interesting, yet simple, pieces of basic economic analysis concerns the incidence of a tax, i.e. identifying who bears the burden of a tax. Public finance theory makes a distinction between the "statutory incidence" of a tax, i.e. who is legally responsible for paying the tax and the "economic incidence" of the tax, i.e. who really bears the burden of the tax.

- This is an area in which economists have often neglected their duty in educating the public: one often reads about discussions in newspapers that assume that the burden of a particular tax would be borne only by consumers or only by firms without careful analysis. For example, suppose the government repeals the sales tax on gasoline, will consumers really benefit from the reduction in the sales tax? Suppose the government enacts a 15% tax on the sale of luxury yachts, who bears the burden, the yacht buyer, the yacht seller, or the worker who helps build the yacht?

- Consider a simple model to think about what factors determine who bears the burden of a tax. For simplicity let us think about a unit tax, a tax that is levied on a particular good as a fixed amount per unit sold. So if P represents the price that the good sells for in the market, the producer only receives $P - T$ for every unit sold. The basic model can then be changed so that

$$\begin{aligned} Q^d &= aY - bP \\ Q^s &= -cW + d(P - T) \\ Q^d &= Q^s \end{aligned}$$

- In matrix form, we can write this model as

$$\begin{bmatrix} 1 & 0 & b \\ 0 & 1 & -d \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} Q^d \\ Q^s \\ P \end{bmatrix} = \begin{bmatrix} aY \\ -dT - cW \\ 0 \end{bmatrix}$$

- The solution to this system can be found by using Cramer's Rule

$$Q^d = \frac{1}{|A|} \begin{vmatrix} aY & 0 & b \\ -dT - cW & 1 & -d \\ 0 & 1 & 0 \end{vmatrix} = \frac{adY - bcW - bdT}{b + d}$$

$$Q^s = \frac{1}{|A|} \begin{vmatrix} 1 & aY & b \\ 0 & -dT - cW & -d \\ -1 & 0 & 0 \end{vmatrix} = \frac{adY - bcW - bdT}{b + d}$$

$$P = \frac{1}{|A|} \begin{vmatrix} 1 & 0 & aY \\ 0 & 1 & -cW - dT \\ -1 & 1 & 0 \end{vmatrix} = \frac{aY + cW + dT}{b + d}$$

- So at a basic level the introduction of a unit tax lowers equilibrium quantity and raises equilibrium price. However, we can dig deeper into the tax incidence question. The impact of the unit tax on the price paid by consumers can be found by calculating $\frac{\partial P}{\partial T} = \frac{d}{b+d}$. The impact of the tax on the price received by firms (which is $P - T$) is given by $\frac{\partial (P-T)}{\partial T} = \frac{d}{b+d} - 1 = \frac{-b}{b+d}$

- So the tax drives down the price that firms receive per unit of goods sold and raises the price paid by consumers. However, the magnitude of this impact depends on the relative size of the parameters b and d.

- If $b > d$, i.e. a change in price has a relatively large effect on the quantity demanded relative to the quantity supplied: i.e. goods with more elastic demand, then producers pay more of the tax burden. On the other hand if $d > b$ i.e. demand is relatively less elastic then consumers pay more of the burden.

- So if we put a unit tax on a good like cigarettes, i.e. a good for which b is close to zero, we would find that $\frac{\partial P}{\partial T} = 1$, and $\frac{\partial (P-T)}{\partial T} = 0$, i.e. consumption would not fall much and smokers bear the burden: good for raising revenue, not so good for discouraging smoking.

- If on the other hand we put a tax on a good for which demand is extremely elastic, say Canadian beer, where b is very large then we would find $\frac{\partial P}{\partial T} = 0$ and $\frac{\partial (P-T)}{\partial T} = 1$, i.e. equilibrium quantity would fall, the price paid by consumers would not change very much while the producers would bear all of the tax burden.

International Trade

- Another useful application to think about is in international trade, where once again there has been a storm of recent protest concerning the impact of globalization and trade liberalization on poor countries. Are these countries really worse off under free trade? Is protectionism the answer?

- We can answer some of these with a simple model along the lines presented in Klein. This simple model is used in trade theory to derive a relationship between input prices and output price known as the "Stolper-Samuelson" effect. The simple model is of an economy that produces two goods: corn (C) and Ferraris (F) using two inputs capital (K) and labor (L).

- Let a_{ij} denote the amount of input i needed to produce one unit of good j. So a_{KF} =capital needed for 1 Ferrari, a_{LF} =labor needed for 1 Ferrari, a_{KC} = capital needed for 1 ton of corn, a_{LC} = capital needed for 1 ton of corn.

- We will also assume that Ferraris are a capital intensive good, while corn is a labor intensive good (i.e. $\frac{a_{KF}}{a_{LF}} > \frac{a_{KC}}{a_{LC}}$)

- We will also denote the rental price paid to a unit of capital as r and the wage of a worker by w. If the market for both goods is competitive (o.k, the Ferrari was not a good example, but I will never be able to buy one so humor me.) then we know that the price of each of the goods will be equal to the marginal cost so that:$r \times a_{KF} + w \times a_{LF} = P_F$ and $r \times a_{KC} + w \times a_{LC} = P_C$

- In matrix form

$$\begin{bmatrix} a_{KF} & a_{LF} \\ a_{KC} & a_{LC} \end{bmatrix} \begin{bmatrix} r \\ w \end{bmatrix} = \begin{bmatrix} P_F \\ P_C \end{bmatrix}$$

- We can then solve to determine the impact of changes in the prices of the goods (under perfect competition firms have to take prices as given exogenously) on the input prices. The solution to the model yields:

$$r = \frac{\begin{vmatrix} P_F & a_{LF} \\ P_C & a_{LC} \end{vmatrix}}{\begin{vmatrix} a_{KF} & a_{LF} \\ a_{KC} & a_{LC} \end{vmatrix}} = \frac{a_{LC}P_F - a_{LF}P_C}{a_{KF}a_{LC} - a_{LF}a_{KC}}$$

$$w = \frac{\begin{vmatrix} a_{KF} & P_F \\ a_{KC} & P_C \end{vmatrix}}{\begin{vmatrix} a_{KF} & a_{LF} \\ a_{KC} & a_{LC} \end{vmatrix}} = \frac{a_{KF}P_C - a_{KC}P_F}{a_{KF}a_{LC} - a_{LF}a_{KC}}$$

- Note that the denominator in both solutions is positive, thanks to the assumption that Ferrari's are more capital intensive.
- An increase in the price of Ferraris raises the rental price of capital and lowers the wage rate. An increase in the price of corn raises the wage rate and lowers the rental price of capital.
- What would the impact of trade be? Well, if we pick a rich country and a poor country, the comparative advantage for the rich country is likely to be in producing Ferraris and the comparative advantage of the poor country is likely to be in producing corn. So free trade would lead to rich countries specializing in Ferrari production and poor countries specializing in corn production.
- As a result, we would expect that rich countries would export Ferraris and import corn while poor countries export corn and import Ferraris (o.k, so these Ferraris really were a bad example).
- In a rich country, we would therefore expect the price of Ferraris to rise and the price of corn to fall with free trade (greater demand for the cars from outside the country and cheap corn available for import.) From our analysis above, we can see that owners of capital would be better off and workers worse off.
- In a poor country, we would expect the price of Ferraris to fall and the price of corn to rise with free trade (greater demand for corn from outside the country and cheap(er) Ferraris available for import.) From our analysis above, we can see that owners of capital would be worse off and workers better off.
- So does free trade worsen the lives of the poor in developing countries? Not according to this model. Free trade hurts two groups: capital owners in poor countries and workers in rich countries, i.e. people who provide the inputs for the good that the country does NOT have a comparative advantage in.
- So we need to be thinking about more sophisticated economic models, if we hope to prove that the arguments of the anti free-trade protestors would indeed hold true in the real world.