LECTURE 10: CONSTRAINED OPTIMIZATION II

I. INTRODUCTION

- In today's class we cover two topics to bring closure to the section on constrained optimization. The first is to derive a very important theorem in Economics known as the **envelope theorem**. This powerful theorem greatly simplifies our ability to do comparative static analysis.

- The second is to consider a very important property of constrained optimization models, known as "duality". Duality states that every constrained maximization/minimization problem (known as the "primal" problem) has an associated minimization/maximization problem involving the constraint (known as the "dual" problem). If you know some key identities, you can always move from the primal to the dual and vice versa.

- The relationship between the primal and the dual also plays a critical role in decomposing the effect of a price change into income and substitution effects, which we will do mathematically in today's lecture and the next lecture.

II. THE ENVELOPE THEOREM

- We begin by deriving a neat result about comparative optimization problems known as the "Envelope Theorem". The envelope theorem simply stated is as follows: when we estimate the impact of changing a parameter/exogenous variable on the optimized value of the objective function we can ignore the impact that such changes in the parameter/exogenous variable have on the optimal choice variables.

- To put it even more simply, after we calculate a solution to an optimization problem, we can do comparative static analysis of how changes in the parameters/exogenous variables affect the optimal value without having to resolve the optimization problem again with the new parameters/exogenous variables.

- The following algebraic formulation should make this clearer. Consider a standard optimization problem of the form (where β is a parameter/exogenous variable of the model)

$$\max_{x_1,\cdots,x_n,\lambda} \mathcal{L} = \max_{x_1,\cdots,x_n,\lambda} f\left(x_1,\cdots,x_n;\beta\right) + \lambda\left(c - g\left(x_1,\cdots,x_n;\beta\right)\right)$$

- We can totally differentiate the Lagrangian as:

$$d\mathcal{L} = \left(\sum_{i=1}^{n}\left[f_i - \lambda g_i\right]dx_i\right) + \left(f_\beta - \lambda g_\beta\right)d\beta + \left[c - g\left(x_1,\cdots,x_n;\beta\right)\right]d\lambda + \lambda dc$$

- The solutions to this problem are derived from the FOC

$$\begin{aligned} f_i(x_1^*, \cdots, x_n^*; \beta) - \lambda^* g_i(x_1^*, \cdots, x_n^*; \beta) &= 0 \\ c - g(x_1^*, \cdots, x_n^*; \beta) &= 0 \end{aligned}$$

- The differential, evaluated at the solution $(x_1^*, \cdots, x_n^*, \lambda^*)$ can be written as

$$\begin{aligned} d\mathcal{L}^* = & \left(\sum_{i=1}^{n} \left[f_i(x_1^*, \cdots, x_n^*; \beta) - \lambda^* g_i(x_1^*, \cdots, x_n^*; \beta) \right] dx_i \right) \\ & + \left[f_\beta(x_1^*, \cdots, x_n^*; \beta) - \lambda^* g_\beta(x_1^*, \cdots, x_n^*; \beta) \right] d\beta + \left[c - g(x_1^*, \cdots, x_n^*; \beta) \right] d\lambda + \lambda^* dc \end{aligned}$$

- While this looks horribly complicated, we can use the FOC to simplify to

$$d\mathcal{L}^* = \left[f_\beta(x_1^*, \cdots, x_n^*; \beta) - \lambda^* g_\beta(x_1^*, \cdots, x_n^*; \beta) \right] d\beta + \lambda^* dc$$

- Since $\mathcal{L}^* = f^*$ we can, therefore, calculate the following: $\frac{df^*}{d\beta} = f_\beta - \lambda^* g_\beta \equiv \mathcal{L}_b eta$ and $\frac{df^*}{dc} = \lambda^* \equiv \mathcal{L}_l ambda$.

- This a very powerful result. It says that impact of a change in an exogenous variable on the maximized objective function can be found by simply taking the partial derivative of the Lagrangian with respect to that exogenous variable, evaluated at the optimal point. In other words, to calculate the impact of a change in the parameters/exogenous variables of the model on the maximized objective function, we don't need to worry about how those changes affect our optimal choices (x_i^*).

- If this is still confusing, think about it this way. What the envelope theorem says is that after we solve an optimization problem, we can gauge how a change in an exogenous variable would affect the optimized value of the objective function without having to resolve the problem using the new value of the exogenous variable.

- Note also that

$$\mathcal{L}^* = f(x_1^*, \cdots, x_n^*; \beta) - \lambda^* \left[c - g(x_1^*, \cdots, x_n^*; \beta) \right] = f(x_1^*, \cdots, x_n^*; \beta)$$

  since $c - g(x_1^*, \cdots, x_n^*; \beta) = 0$ is one of the FOCs. In other words, $\frac{d\mathcal{L}^*}{d\beta} \equiv \frac{df^*}{d\beta}$, or in words, the impact on the maximized Lagrangian of changing a parameter is the same as its impact on the maximized objective function, since the maximized Lagrangian and the maximized objective function are the same thing.

Example 1: Interpreting the Lagrange Multiplier

- Consider the total differential of the maximized Lagrangian

$$d\mathcal{L}^* = \left[f_\beta(x_1^*, \cdots, x_n^*; \beta) - \lambda^* g_\beta(x_1^*, \cdots, x_n^*; \beta) \right] d\beta + \lambda^* dc$$

- If we ignore changes in all other exogenous variables, this reduces to $d\mathcal{L}^* = \lambda^* dc$ or $\frac{d\mathcal{L}^*}{dc} = \lambda^*$.

- In other words, we can describe the solution for the Lagrange multiplier as the amount by which a change in the constraint affects the optimal value of the objective function. Note that this interpretation came about because we wrote the constraint as $c - g(x_1, \cdots, x_n) = 0$ instead of $g(x_1, \cdots, x_n) - c = 0$. Writing the constraint in the latter style, while generating the same solution will imply for an interpretation of -λ, not of λ.

Example 2: Utility Maximization

- Consider the following utility maximization problem

$$\max_{C_1, C_2} \sqrt{C_1 C_2} \text{ subject to } P_1 C_1 + P_2 C_2 = Y$$

- The Lagrangian function for this problem is

$$\mathcal{L} = 2\sqrt{C_1 C_2} + \lambda(Y - P_1 C_1 - P_2 C_2)$$

- We can derive the solutions $C_1^* = \frac{Y}{2P_1}$, $C_2^* = \frac{Y}{2P_2}$, and $\lambda^* = \frac{1}{2\sqrt{P_1 P_2}}$.
- Now suppose we were thinking about what would happen if we relax the constraint a little, e.g. increase Y by \$1. What the envelope theorem tells us is that we can calculate that by simply taking the partial derivative of the Lagrangian with respect to Y (evaluated at the optimum):

$$\frac{\partial U^*}{\partial Y} = \frac{\partial \mathcal{L}}{\partial Y} = \lambda^* = \frac{1}{2\sqrt{P_1 P_2}}$$

- Similarly, we can think about what would happen to maximized utility if the price of good 1 were to increase by a dollar. The envelope theorem tells us that we can calculate that by simply taking the partial derivative of the Lagrangian with respect to P_1 (evaluated at the optimum):

$$\frac{\partial U^*}{\partial P_1} = \frac{\partial \mathcal{L}}{\partial P_1} = -\lambda^* C_1^* = -\left(\frac{1}{2\sqrt{P_1 P_2}}\right)\left(\frac{Y}{2P_1}\right) = -\left(\frac{Y}{4P_1^{3/2} P_2^{1/2}}\right)$$

- We can verify these answers by calculating that the optimal level of utility that this consumer can achieve, given the constraints is a utility level of $U^* = \sqrt{\frac{Y^2}{(2P_1)(2P_2)}} = \frac{Y}{2\sqrt{P_1 P_2}}$
- You can see that $\frac{\partial U^*}{\partial Y}$ is indeed equal to $\frac{1}{2\sqrt{P_1 P_2}}$ and that $\frac{\partial U^*}{\partial P_1}$ is indeed equal to $-\left(\frac{Y}{4P_1^{3/2} P_2^{1/2}}\right)$

III. DUALITY THEORY

- As stated in the introduction, every constrained optimization problem has the wonderful property of duality, which implies that the maximization/minimization problem (known as the "primal" problem) has an associated minimization/maximization problem involving the constraint (known as the "dual" problem).
- We can best illustrate this with a standard utility maximization example from consumer theory.

The Primal problem

- Consider the following simple model where you maximize $U(x_1, x_2)$ subject to $Y = P_1x_1 + P_2x_2$. The solutions to this model yield demand functions for the two goods.

$$\begin{aligned} C_1^* &= D_1(P_1, P_2, Y) \\ C_2^* &= D_2(P_1, P_2, Y) \end{aligned}$$

- These demand functions, where quantity demanded is a function of prices and income, are known as Marshallian demand functions, to distinguish them from another way of expressing the demand function, known as the Hicksian demand function.
- Marshallian demand curves are what we traditionally think of as demand curves in economics - they show how the quantity of goods demanded by an individual is affected by changes in price, holding prices of other goods and income constant.
- We can plug these solutions back into the utility function and derive an expression for maximized utility as

$$U^* = V(P_1, P_2, Y)$$

- The V function is known as the indirect utility function - it tells the highest level of utility that can be attained for a given income level and price levels of the good.

The Dual problem

- Since the primal problem was a maximization, the dual problem is a minimization involving the constraint. In this case, that problem is to minimize $P_1x_1 + P_2x_2$ subject to the constraint that $U(x_1, x_2) = \bar{U}$. In other words the dual of a utility maximization is an expenditure minimization problem, which calculates the lowest cost way of attaining a particular level of utility.
- The solutions to this model yield demand functions for the two goods.

$$\begin{aligned} C_1^* &= H_1(P_1, P_2, \bar{U}) \\ C_2^* &= H_2(P_1, P_2, \bar{U}) \end{aligned}$$

- These demand functions, where quantity demanded is a function of prices and a given utility level, are known as Hicksian demand functions, or alternatively as compensated demand functions.

- Hicksian demand curves are important in economics because they tell us how the quantity demanded changes with price, holding utility constant. In other words, Hicksian demand curves show us how the consumer's optimal bundle moves along her original indifference curve when there is a price change - i.e. it is tracking the substitution effect.

- We can plug these solutions back into the objective function and derive an expression for minimized expenditure as
$$E^* = E(P_1, P_2, \bar{U})$$

How the Primal is related to the Dual

- The relationship between the primal and the dual can be defined very simply as follows: when $\bar{U} = U^*$ then $E^* = Y$. In other words, the minimum expenditure needed to achieve the optimized level of utility has to equal the income of the consumer. It can't be greater or else the budget constraint is violated, and it cannot be less because then the consumer is not maximizing utility (she should be able to spend more on goods that enhance her utility).

- We can summarize the relationship between Marshallian and Hicksian demand with the following relationships

$$\begin{aligned} H_i(P_1, P_2, V(P_1, P_2, Y)) &= D_i(P_1, P_2, Y) \text{for } \forall i \\ H_i(P_1, P_2, U^*) &= D_i(P_1, P_2, E(P_1, P_2, U^*)) \text{for } \forall i \end{aligned}$$

- In other words, the Hicksian demand curve and the Marshallian demand curve are identical if you evaluate the Hicksian demand curve at the optimized level of utility and the Marshallian demand curve at the minimized level of expenditure needed to achieve the optimized level of utility.

IV. MOVING SEAMLESSLY BETWEEN MARSHALLIAN DEMAND AND HICKSIAN DEMAND

- Now that we know how the primal and the dual are related, we should be able to switch from the expenditure function to the indirect utility function and vice versa.

- The more important challenge is to learn how to go from Marshallian demand curves to Hicksian demand curves and vice versa. To do that we need two relationships *Roy's Identity*, which allows us to recover Marshallian demand from indirect utility and *Shephard's Lemma*, which allows us to recover Hicksian demand from the expenditure function.

- I will first derive these relationships (both of which are applications of the envelope theorem, and then show how you would move from Marshallian demand curves to Hicksian demand curves and vice versa.

Roy's Identity

- Consider the primal problem Max $U(x_1, x_2)$ subject to $Y = P_1 x_1 + P_2 x_2$. The Lagrangian for the primal problem is

$$\mathcal{L} = U(x_1, x_2) + \lambda(Y - P_1 x_1 - P_2 x_2)$$

- Using the envelope theorem we know that $\frac{dU^*}{dP_1} = \frac{\partial \mathcal{L}}{\partial P_1} \equiv -\lambda^* x_1$ and that $\frac{dU^*}{dP_1} = \frac{\partial \mathcal{L}}{\partial P_1} \equiv \lambda$.
- Combining, we obtain $x_1 = \frac{-\frac{dU^*}{dP_1}}{\frac{dU^*}{dY}} \equiv \frac{-\frac{dV}{dP_1}}{\frac{dV}{dY}}$.
- This relationship is known as Roy's identity

$$D_1(P_1, P_2, Y) = \frac{-\frac{dV(P_1,P_2,Y)}{dP_1}}{\frac{dV(P_1,P_2,Y)}{dY}}$$

- Roy's identity is useful because it allows us to retrieve Marshallian demand curves from the indirect utility function.

Shephard's Lemma

- Now consider the dual problem Min $P_1x_1 + P_2x_2$ subject to $U(x_1, x_2) = \bar{U}$. The Lagrangian for the dual problem is

$$\mathcal{L} = P_1x_1 + P_2x_2 + \lambda(\bar{U} - U(x_1, x_2))$$

- Using the envelope theorem we know that $\frac{dE^*}{dP_1} = \frac{\partial \mathcal{L}}{\partial P_1} \equiv x_1$
- This relationship is called Shephard's Lemma

$$H_1(P_1, P_2, U^*) = \frac{dE(P_1, P_2, U^*)}{dP_1}$$

- Shephard's Lemma is a useful result because it let's us retrieve Hicksian demand curves from the expenditure function.

V. AN EXAMPLE TO ILLUSTRATE EVERYTHING

- Consider the following maximization problem maximize $U(x_1, x_2) = x_1^(1/2)x_2^(1/2)$ subject to $Y = P_1x_1 + P_2x_2$. With a little bit of algebra you can show that the solutions to this model yield demand functions for the two goods.

$$\begin{aligned} x_1^* &= D_1(P_1, P_2, Y) \equiv \frac{Y}{2P_1} \\ x_2^* &= D_2(P_1, P_2, Y) \equiv \frac{Y}{2P_2} \end{aligned}$$

- We can plug these solutions back into the utility function and derive an expression for maximized utility as

$$U^* = V(P_1, P_2, Y) \equiv \frac{Y}{2(P_1P_2)^{1/2}}$$

- The corresponding dual problem is to minimize $P_1x_1 + P_2x_2$ subject to $\bar{U} = x_1^{1/2}x_2^{1/2}$. With a little bit of algebra you can show that the solutions to this model yield demand functions for the two goods.

$$x_1^* = H_1(P_1, P_2, \bar{U}) \equiv \bar{U}\left(\frac{P_2}{P_1}\right)^{1/2}$$
$$x_2^* = H_2(P_1, P_2, \bar{U}) \equiv \bar{U}\left(\frac{P_1}{P_2}\right)^{1/2}$$

- We can plug these solutions back into the objective function and derive an expression for minimized expenditure as

$$E^* = E(P_1, P_2, \bar{U}) \equiv 2\bar{U}(P_1 P_2)^{1/2}$$

Matching Up the Dual With the Primal

- You can now see the correspondence between the primal and the dual. When we evaluate the expenditure function at $\bar{U} = U^* \equiv \frac{Y}{2(P_1 P_2)^{1/2}}$ we get

$$E(P_1, P_2, U^*) = 2\left(\frac{Y}{2(P_1 P_2)^{1/2}}\right)(P_1 P_2)^{1/2} = Y$$

- We can show the equivalence between Hicksian and Marshallian demand as

$$H_1(P_1, P_2, U^*) \equiv U^*\left(\frac{P_2}{P_1}\right)^{1/2} = \left(\frac{Y}{2(P_1 P_2)^{1/2}}\right)\left(\frac{P_2}{P_1}\right)^{1/2}$$
$$= \frac{Y}{2P_1} \equiv D_1(P_1, P_2, Y)$$

$$H_2(P_1, P_2, U^*) \equiv U^*\left(\frac{P_1}{P_2}\right)^{1/2} = \left(\frac{Y}{2(P_1 P_2)^{1/2}}\right)\left(\frac{P_1}{P_2}\right)^{1/2}$$
$$= \frac{Y}{2P_2} \equiv D_2(P_1, P_2, Y)$$

Moving from the expenditure function to the indirect utility function

- Finally, we can show how to move from the expenditure function to the indirect utility function and vice versa. We can rearrange the expenditure function $E^* = 2U^*(P_1 P_2)^{1/2}$ to get $U^* = \frac{E^*}{2(P_1 P_2)^{1/2}}$. Since $E^* = Y$ we can see that this is in fact the indirect utility function $V(P_1, P_2, Y) = \frac{Y}{2(P_1 P_2)^{1/2}}$

- Similarly, we can rearrange the indirect utility function $U^* = \frac{Y}{2(P_1 P_2)^{1/2}}$ to get $Y = 2U^*(P_1 P_2)^{1/2}$. Since $Y = E^*$, we can see that this is in fact the expenditure function $E(P_1, P_2, U^*) = 2U^*(P_1 P_2)^{1/2}$

Moving from Marshallian Demand to Hicksian Demand

- Consider the two Marshallian demand curves we obtained

$$x_1^* = \frac{Y}{2P_1}$$
$$x_2^* = \frac{Y}{2P_2}$$

- We can plug these into the utility function and obtain the indirect utility function $U^* = V(P_1, P_2, Y) \equiv \frac{Y}{2(P_1P_2)^{1/2}}$. This can be transformed into the expenditure function by rearranging terms, and using the fact that $Y = E^*$ at maximized utility U^*. So we have

$$E^* \equiv E(P_1, P_2, U^*) = 2U^*(P_1P_2)^{1/2}$$

- Applying Shephard's lemma gives us the Hicksian demands

$$H_1(P_1, P_2, U^*) = \frac{dE(P_1, P_2, U^*)}{dP_1} \equiv U^* \left(\frac{P_2}{P_1}\right)^{1/2}$$
$$H_2(P_1, P_2, U^*) = \frac{dE(P_1, P_2, U^*)}{dP_2} \equiv U^* \left(\frac{P_1}{P_2}\right)^{1/2}$$

which is what we derived before.

Moving from Hicksian Demand to Marshallian Demand

- Consider the two Hicksian demand curves we obtained, evaluated at U^*

$$H_1(P_1, P_2, U^*) \equiv U^* \left(\frac{P_2}{P_1}\right)^{1/2}$$
$$H_2(P_1, P_2, U^*) \equiv U^* \left(\frac{P_1}{P_2}\right)^{1/2}$$

- We can plug these solutions back into the objective function and derive an expression for minimized expenditure as $E^* = 2U^*(P_1P_2)^{1/2}$. This can be transformed into the indirect utility function by rearranging terms, and using the fact that $Y = E^*$ at maximized utility U^*. So we have

$$U^* = V(P_1, P_2, Y) \equiv \frac{Y}{2(P_1P_2)^{1/2}}$$

- Applying Roy's identity gives us the Marshallian demands

$$
\begin{aligned}
D_1(P_1, P_2, U^*) &= \frac{-\frac{\partial V(P_1,P_2,Y)}{\partial P_1}}{\frac{\partial V(P_1,P_2,Y)}{\partial Y}} = \frac{-\left(\frac{-Y}{4(P_1^{3/2}P_2^{1/2})}\right)}{\left(\frac{1}{2(P_1^{1/2}P_2^{1/2})}\right)} = \frac{Y}{2P_1} \\
D_2(P_1, P_2, U^*) &= \frac{-\frac{\partial V(P_1,P_2,Y)}{\partial P_2}}{\frac{\partial V(P_1,P_2,Y)}{\partial Y}} = \frac{-\left(\frac{-Y}{4(P_1^{1/2}P_2^{3/2})}\right)}{\left(\frac{1}{2(P_1^{1/2}P_2^{1/2})}\right)} = \frac{Y}{2P_2}
\end{aligned}
$$

which is what we derived before.