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LECTURE 12: MODELS OF STRATEGIC INTERACTION

I. INTRODUCTION

- In today's class we bring closure to the section on optimization by discussing an important class of problems where the optimization decision of one economic agent, depends on the optimization decision of another agent (and vice versa). There are several different ways to handle this interrelationship, which we call "conjectural variation".
- We will look at three commonly used method - Cournot competition, where each party treats the others quantity decision as given in making their own quantity decision, Bertrand competition, which is similar to Cournot in terms of how each party treats the actions of the other, except that the decisions are made about price instead of quantity, and Stackelberg competition where one party (the leader) has the ability to influence the decisions of the other party (the follower).

II. OPTIMIZATION PROBLEMS WITH INTERACTION

- Optimization models where one decision maker's choice affects the other's decision are common in game theory and in many other areas of economics.
- Consider a duopoly market, i.e. a market which has two firms: firm A and firm B, producing the same product, with both firms being price setters. Each firm's decisions about what price to set, and how much to produce, affects the pricing and production decisions of the other. So in essence the two firms are in direct competition with each other in this market.
- There are two ways in which economists solve complicated problems of market competition. They assume either that the two firms are engaged in price competition or that the firms are engaged in quantity competition. In other words both firms are assumed to compete either by picking the price that maximizes their profits given the other firm's choice of price or by picking the production quantity that maximizes profits given the other firm's choice of quantity.
- Quantity competition along these lines is known as **Cournot** competition or **Stackelberg** competition depending on certain assumptions we make about strategic behavior, while price competition is known as **Bertrand** competition.
- In these type of models we have to worry about expectations: what party A thinks party B will do can influence party A's decision. On the other hand party A's decision may influence party B's decision, which may be something that party A needs to take into account in making her decision.
- Suppose the two firms A and B, have respective cost functions $C_A(Q_A) = Q_A^2 + aQ_A + F_A$ and $C_B(Q_B) = Q_B^2 + bQ_B + F_B$. We can interpret F_A and F_B as fixed costs for the two firms and a and b as being parameters that affect the marginal costs for firms A and B respectively.

- Suppose the demand curve for the good is given by $P(Q) = \alpha - \beta Q$ where $Q = Q_A + Q_B$. We can interpret the parameter α as a demand shifter (i.e. an increase in α shifts out the demand curve) and the parameter β as a slope changer (i.e. an increase in β changes the slope of the demand curve). An example of α may be a change in overall income of consumers (which would increase demand); an example of β may be the availability of a substitute good which may make consumers more responsive to price changes.

- The maximization problem for the two producers are

$$\begin{aligned}
\max_{Q_A} \Pi_A(Q_A) &= P(Q)Q_A - C_A(Q_A) \equiv [\alpha - \beta(Q_A + Q_B)]\, Q_A - \left(Q_A^2 + aQ_A + F_A\right) \\
\max_{Q_B} \Pi_B(Q_B) &= P(Q)Q_B - C_B(Q_B) \equiv [\alpha - \beta(Q_A + Q_B)]\, Q_B - \left(Q_B^2 + bQ_B + F_B\right)
\end{aligned}$$

- The first order conditions for each producer's maximization decision can then be calculated.A

$$\begin{aligned}
\left[\alpha - 2\beta Q_A - \beta Q_B - \beta Q_A \left(\frac{\partial Q_B}{\partial Q_A}\right)\right] - (2Q_A + a) &= 0 \\
\left[\alpha - 2\beta Q_B - \beta Q_A - \beta Q_B \left(\frac{\partial Q_A}{\partial Q_B}\right)\right] - (2Q_B + b) &= 0
\end{aligned}$$

- The terms $\left(\frac{\partial Q_B}{\partial Q_A}\right)$ and $\left(\frac{\partial Q_A}{\partial Q_B}\right)$ are called **conjectural variation** terms. They represent each firm's guess (or conjecture) about what the other firm will do, and how their own decision will affect the other firm's behavior.

- We will look at some common assumptions about the conjectural variation term next.

Cournot Competition

- In the case of **Cournot competition**, the assumption is that each firm takes the other's decision as given (i.e. uninfluenceable) in making her decision. Thus, producer A treats Q_B as fixed in making her decision and producer B treats Q_A is fixed in making her decision. This means that the conjectural variation terms are zero, and the system of FOCs becomes

$$\begin{aligned}
[\alpha - 2\beta Q_A - \beta Q_B] - (2Q_A + a) &= 0 \\
[\alpha - 2\beta Q_B - \beta Q_A] - (2Q_B + b) &= 0
\end{aligned}$$

- This can be represented in matrix form

$$\begin{bmatrix} 2(1+\beta) & \beta \\ \beta & 2(1+\beta) \end{bmatrix} \begin{bmatrix} Q_A \\ Q_B \end{bmatrix} = \begin{bmatrix} \alpha - a \\ \alpha - b \end{bmatrix}$$

- The solution to this system will give us the optimal quantities chosen. using Cramer's Rule we can derive the following:

$$Q_A = \frac{\begin{vmatrix} \alpha - a & \beta \\ \alpha - b & 2(1+\beta) \end{vmatrix}}{\begin{vmatrix} 2(1+\beta) & \beta \\ \beta & 2(1+\beta) \end{vmatrix}}, Q_B = \frac{\begin{vmatrix} 2(1+\beta) & \alpha - a \\ \beta & \alpha - b \end{vmatrix}}{\begin{vmatrix} 2(1+\beta) & \beta \\ \beta & 2(1+\beta) \end{vmatrix}}$$

- The solutions are $Q_A = \frac{(2+\beta)\alpha+(\beta)b-2(1+\beta)a}{(2+3\beta)(2+\beta)}$ and $Q_B = \frac{(2+\beta)\alpha+(\beta)a-2(1+\beta)b}{(2+3\beta)(2+\beta)}$.
- We can also verify the SOC. Since these are individual optimization problems, we would calculate the 2nd derivatives instead of the Hessian. $\frac{\partial^2 \Pi_A}{\partial Q_A^2} = -2(1+\beta) < 0$ and $\frac{\partial^2 \Pi_B}{\partial Q_B^2} = -2(1+\beta) < 0$ indicating that the solution to the FOC indeed is a maximum.
- The answers make intuitive sense: an increase in a, which increases Firm A's marginal costs, increase firm B's quantity and decreases firm A's quantity. So firm B gains market share. The converse holds for an increase in b. Finally an increase in α, which represents an exogenous increase in the quantity demanded, raises both firms' production equally.

Stackelberg Competition

- Consider a slight variant of the duopoly market described above. Instead of assuming that the two firms take each others quantity decisions as given, we instead consider firm A to be a "leader" firm and firm B to be a "follower" firm.
- In other words, firm B takes firm A's output as given but firm A is fully cognizant of firm B's production when making its own decision.
- In terms of the conjectural variation terms, we have $\frac{\partial Q_A}{\partial Q_B} = 0$ and $\frac{\partial Q_B}{\partial Q_A} \neq 0$
- The system of FOCs becomes

$$\begin{aligned} \left[\alpha - 2\beta Q_A - \beta Q_B - \beta Q_A \left(\frac{\partial Q_B}{\partial Q_A}\right)\right] - (2Q_A + a) &= 0 \\ [\alpha - 2\beta Q_B - \beta Q_A] - (2Q_B + b) &= 0 \end{aligned}$$

- Using the implicit function theorem, we can calculate that

$$\frac{\partial Q_B}{\partial Q_A} = \frac{-(-\beta)}{-2\beta - 2} = \frac{-\beta}{2(\beta+1)}$$

- The first order conditions then become

$$\begin{aligned} \left[\alpha - 2\beta Q_A - \beta Q_B - \beta Q_A \left(\frac{-\beta}{2(\beta+1)}\right)\right] - (2Q_A + a) &= 0 \\ [\alpha - 2\beta Q_B - \beta Q_A] - (2Q_B + b) &= 0 \end{aligned}$$

- We can set up the FOCs as a system of equations represented in matrix form

$$\begin{bmatrix} \left(\frac{4+8\beta+3\beta^2}{2(1+\beta)}\right) & \beta \\ \beta & 2(1+\beta) \end{bmatrix} \begin{bmatrix} Q_A \\ Q_B \end{bmatrix} = \begin{bmatrix} \alpha - a \\ \alpha - b \end{bmatrix}$$

- The solution to this system will give us the optimal quantities chosen. Using Cramer's Rule we get

$$Q_A = \frac{\begin{vmatrix} \alpha - a & \beta \\ \alpha - b & 2(1+\beta) \end{vmatrix}}{\begin{vmatrix} \left(\frac{4+8\beta+3\beta^2}{2(1+\beta)}\right) & \beta \\ \beta & 2(1+\beta) \end{vmatrix}}, Q_B = \frac{\begin{vmatrix} \left(\frac{4+8\beta+3\beta^2}{2(1+\beta)}\right) & \alpha - a \\ \beta & \alpha - b \end{vmatrix}}{\begin{vmatrix} \left(\frac{4+8\beta+3\beta^2}{2(1+\beta)}\right) & \beta \\ \beta & 2(1+\beta) \end{vmatrix}}$$

- The solutions are $Q_A = \frac{(2+\beta)\alpha - 2(1+\beta)a + (\beta)b}{4+8\beta+2\beta^2}$ and $Q_B = \frac{\left(\frac{4+6\beta+\beta^2}{2(1+\beta)}\right)\alpha + (\beta)a - \left(\frac{4+8\beta+3\beta^2}{2(1+\beta)}\right)b}{4+8\beta+2\beta^2}$.
- The answers make intuitive sense: an increase in a, which increases Firm A's marginal costs, increase firm B's quantity and decreases firm A's quantity. So firm B gains market share. When we consider an increase in b we see that this will increase the output produced by firm A and lower the output produced by firm B.
- We can also verify the SOC. Again since these are individual optimization problems, we would calculate the 2nd derivatives instead of the Hessian. $\frac{\partial^2 \Pi_A}{\partial Q_A^2} = -2(1+\beta) + \frac{\beta^2}{2(1+\beta)} < 0$ and $\frac{\partial^2 \Pi_B}{\partial Q_B^2} = -2(1+\beta) < 0$ indicating that the solution to the FOC indeed is a maximum.

Bertrand Competition

- Another variant of the duopoly market is when we assume that firms compete over price, instead of quantity.
- Under Bertrand competition, each firm will produce at the quantity such that P=MC. Why? First of all, since the two goods are identical, they have to sell for the same price.
- Second, if that price is higher than the marginal cost of one of the firms, that firm will have incentive to cut prices, trying to steal market share away from the other firm, pushing price lower towards MC.
- The marginal cost function for the two firms are $P = MC_A = 2Q_A + a$ and $P = MC_B = 2Q_B + b$ respectively. Therefore we get the following results: $Q_A = \frac{P-a}{2}$ and $Q_B = \frac{P-b}{2}$. Combining, we get the aggregate quantity produced as being: $Q = Q_A + Q_B = \frac{2P-a-b}{2}$.
- Plugging this into the demand curve, $P = \alpha - \beta Q$, we get that $P = \alpha - \beta\left(\frac{2P-a-b}{2}\right)$
- This can be solved to get

$$P = \frac{2\alpha + \beta(a+b)}{2(1+\beta)}$$

- We can then back out the solutions for Q_A and Q_B as

$$Q_A = \frac{P-a}{2} = \frac{2\alpha - (2+\beta)a + \beta b}{2+2\beta} \text{ and } Q_B = \frac{P-b}{2} = \frac{2\alpha - (2+\beta)b + \beta a}{2+2\beta}$$