Fall Semester '05-'06
Akila Weerapana

LECTURE 18: Systems of Equations

I. INTRODUCTION

- In the last lecture we discussed how to calculate the eigenvalues and eigenvectors of a matrix. When the eigenvectors of a matrix were linearly independent, the matrix was said to be diagonalizable. We then discussed how to use the eigenvectors and eigenvalues to decompose a matrix into a product of 3 simpler matrices involving eigenvalues and eigenvectors.
- Decomposing the matrix in this manner has several useful economic applications. The most important application is that the decomposition allows us to simplify a multivariate system of linear difference equations into a system of univariate difference equations. So eigenvalues and eigenvectors are extremely important in solving systems of linear multivariate difference equations.
- Today's lecture discusses how to use eigenvalues and eigenvectors to solve systems of multivariate difference and differential equations as well as solving higher order univariate linear difference and differential equations.

II. SOLVING SYSTEMS OF FIRST ORDER LINEAR DIFFERENCE EQUATIONS

- A **system** of first order linear difference equations is of the form $X_t = AX_{t-1} + b$ where X is an $n \times 1$ vector of discretely changing variables, A is an $n \times n$ square matrix and b is an $n \times 1$ vector of constants (time invariant exogenous variables).
- Our analysis of how to solve difference equations did not cover multivariate difference equations. Now we have to solve not just one but a system of multivariate difference equations. We can simplify the problem, however, into a much simpler one by using the diagonalization of the matrix A. IF A is diagonalizable, then the system can be written as

$$\begin{aligned} X_t &= AX_{t-1} + b \\ X_t &= C\Lambda C^{-1}X_{t-1} + b \\ C^{-1}X_t &= \Lambda C^{-1}X_{t-1} + C^{-1}b \end{aligned}$$

- Defining a new variable $Z_t = C^{-1}X_t$ we can express the system in terms of Z as $Z_t = \Lambda Z_{t-1} + C^{-1}b$. Since Λ is diagonal, this is now a collection of univariate difference equations, which we already know how to solve.
- Once we solve for Z, we can easily derive the solution for X by calculating $Z_t = C^{-1}X_t \Rightarrow X_t = CZ_t$.

Example

- A simple example of such a system would be the following

$$\begin{aligned} x_t &= 2x_{t-1} + 2y_{t-1} + 6 \\ y_t &= 2x_{t-1} - y_{t-1} + 12 \end{aligned}$$

- We are also given initial values $x_0 = -3$ and $y_0 = 4$. We can rewrite this system in matrix form as

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

- So in this case $X_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix}$, $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$.

- We need to first find the eigenvalues and eigenvectors of A. We already showed earlier that the eigenvalues of A are $\lambda = 3$ and $\lambda = -2$, with corresponding eigenvectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

- The decomposition of A is then $A = C\Lambda C^{-1}$ where $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$, and $C^{-1} = \begin{bmatrix} 2/5 & 1/5 \\ 1/5 & -2/5 \end{bmatrix}$.

- If we define $Z_t = C^{-1}X_t \equiv \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix}$, the system of difference equations is now $Z_t = \Lambda Z_{t-1} + C^{-1}b$, which is a simple system of univariate difference equations of the form

$$\begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} 2/5 & 1/5 \\ 1/5 & -2/5 \end{bmatrix} \begin{bmatrix} 6 \\ 12 \end{bmatrix} \equiv \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} 24/5 \\ -18/5 \end{bmatrix}$$

- The solutions to the two univariate difference equations $z_{1t} = 3z_{1t-1} + 24/5$ and $z_{2t} = -2z_{2t-1} - 18/5$ can be easily calculated as $z_{1t} = D(3)^t - 12/5$, and $z_{2t} = E(-2)^t - 6/5$ using techniques we developed in the section on difference equations.

- We can then back out the solutions for X_t using the fact that $Z_t = C^{-1}X_t \Rightarrow X_t = CZ_t$. Therefore

$$X_t = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} D(3)^t - 12/5 \\ E(-2)^t - 6/5 \end{bmatrix} \equiv \begin{bmatrix} 2D(3)^t + E(-2)^t - 6 \\ D(3)^t - 2E(-2)^t \end{bmatrix}$$

- The initial conditions can be used to calculate $x_0 \equiv -3 = 2D + E - 6$ and $y_0 \equiv 4 = D - 2E$, which simplifies to $D = 2$ and $E = -1$.

- The solutions to the system of difference equations is

$$\begin{aligned} x_t &= 4(3)^t - (-2)^t - 6 \\ y_t &= 2(3)^t + 2(-2)^t \end{aligned}$$

  You can verify that this is indeed a solution to the system of difference equations we started with.

Dynamics

- The steady states of the system of difference equations $X_t = AX_{t-1} + b$ can be found by setting $X_t = X_{t-1} \equiv X^*$. We get $X^* = AX^* + b \Rightarrow X^* = [I - A]^{-1}b$.
- Consider the above example. From the solutions we see that the eigenvalues of A are what appear in the terms raised to the power t. So the stability of the system depends on the eigenvalues.
- If all the eigenvalues are less than 1 in absolute value then the system is said to be **stable**. If all the eigenvalues are greater than 1 in absolute value then the system is **unstable**. If at least one eigenvalue is less than 1 in absolute value the system is said to be **neutrally stable** or **saddle-path stable**.

III. SOLVING HIGHER ORDER LINEAR DIFFERENCE EQUATIONS

- We can also use the technique for solving a system of first-order difference equations to solve a higher order difference equation. I will illustrate using a second order difference equation, the method easily generalizes for an nth order difference equation.
- Consider a 2nd order difference equation of the form $x_t = \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \beta$. First, we will define an artificial variable $y_t = x_{t-1}$. Then the single second order difference equation can be written as a system of first order difference equations of the form

$$\begin{aligned} x_t &= \alpha_1 x_{t-1} + \alpha_2 y_{t-1} + \beta \\ y_t &= x_{t-1} \end{aligned}$$

- Since this is a system of linear first order difference equations. we can then use the techniques from the last class to calculate a solution to the system, and by extension to the second order difference equation.
- This technique generalizes easily to an nth order difference equation. We first define $(n - 1)$ artificial variables, each one conforming to a particular lagged value of the dependent variable. We then express the nth order difference equation as a system of n equations, the first being a first order difference equation in n variables and the remaining $(n-1)$ being identity equations.
- In order to find an exact solution we also need n starting values, not just a single initial starting value.

Example

- Consider the following 2nd order difference equation: $x_t + \frac{1}{2}x_{t-1} - \frac{1}{2}x_{t-2} = 5$ with initial conditions $x_0 = 8$ and $x_{-1} = 8$.
- By defining an artificial variable $y_t = x_{t-1}$, the second order difference equation can be written as a system of first order difference equations of the form

$$\begin{aligned} x_t &= -\frac{1}{2}x_{t-1} + \frac{1}{2}y_{t-1} + 5 \\ y_t &= x_{t-1} \end{aligned}$$

- In matrix form, we have a system of equations of the form

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

- What we have is a system of first order linear difference equations of the form $X_t = AX_{t-1}+b$. To solve this system, we first need to find the eigenvalues and eigenvectors of A.

- The eigenvalues are the values of λ that solve $\begin{vmatrix} -1/2-\lambda & 1/2 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + \lambda/2 - 1/2 = 0$ which are $\lambda = -1$ and $\lambda = 1/2$.

- The eigenvectors corresponding to $\lambda = -1$ satisfy $\begin{bmatrix} 1/2 & 1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, i.e. $\begin{bmatrix} a \\ -a \end{bmatrix}$.

- The eigenvectors corresponding to $\lambda = 1/2$ satisfy $\begin{bmatrix} -1 & 1/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, i.e. $\begin{bmatrix} a \\ 2a \end{bmatrix}$.

- Since the eigenvectors are linearly independent A is diagonalizable. The decomposition of A is then $A = C\Lambda C^{-1}$ where $\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 1/2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ and the inverse of C can be calculated as $C^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix}$.

- Define $Z_t = C^{-1}X_t$, which gives us a system $Z_t = \Lambda Z_{t-1} + C^{-1}b$ We need to solve this simple system of univariate difference equations of the form

$$\begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix} + \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} 10/3 \\ 5/3 \end{bmatrix}$$

- The solutions to the two univariate difference equations $z_{1t} = -z_{1t-1} + 10/3$ and $z_{2t} = 1/2 z_{2t-1} + 5/3$ can be easily calculated as $z_{1t} = D(-1)^t + 5/3$, and $z_{2t} = E(1/2)^t + 10/3$ using techniques we developed in the section on difference equations.

- We can then back out the solutions for Z_t using the fact that $Z_t = C^{-1}X_t \Rightarrow X_t = CZ_t$. Therefore

$$X_t \equiv \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} D(-1)^t + 5/3 \\ E\left(\frac{1}{2}\right)^t + 10/3 \end{bmatrix} \equiv \begin{bmatrix} D(-1)^t + E\left(\frac{1}{2}\right)^t + 5 \\ -D(-1)^t + 2E\left(\frac{1}{2}\right)^t + 5 \end{bmatrix}$$

- The initial conditions can be used to calculate $x_0 \equiv 8 = D + E + 5$ and $y_0 \equiv x_{-1} \equiv 8 = -D + 2E + 5$, which simplifies to $D = 1$ and $E = 2$.

- The solutions to the system of difference equations is

$$\begin{aligned} x_t &= (-1)^t + 2\left(\frac{1}{2}\right)^t + 5 \\ y_t &= -(-1)^t + 4\left(\frac{1}{2}\right)^t + 5 \end{aligned}$$

- You can verify that this is indeed a solution to the system of difference equations we started with. You can also verify that $y_t = x_{t-1}$, which should hold since we defined the artificial variable y_t this way.

IV. SOLVING SYSTEMS OF FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

- We can use the exact same technique to solve a system of first order linear **differential** equations of the form $\dot{X}_t = AX_t + b$ where X is an $n \times 1$ vector of continuously changing variables, A is an $n \times n$ square matrix and b is an $n \times 1$ vector of constants (time invariant exogenous variables).
- If A is diagonalizable, then the system can be written as

$$\begin{aligned}\dot{X}_t &= AX_t + b \\ \dot{X}_t &= C\Lambda C^{-1}X_t + b \\ C^{-1}\dot{X}_t &= \Lambda C^{-1}X_t + C^{-1}b\end{aligned}$$

- Defining a new variable $Z_t = C^{-1}X_t \Rightarrow \dot{Z}_t = C^{-1}\dot{X}_t$ we can express the system in terms of Z as $\dot{Z}_t = \Lambda Z_t + C^{-1}b$. Since Λ is diagonal, this is now a collection of univariate differential equations, which we already know how to solve.
- Once we solve for Z, we can easily derive the solution for X by calculating $Z_t = C^{-1}X_t \Rightarrow X_t = CZ_t$.

Example

- A simple example of such a system would be the following

$$\begin{aligned}\dot{x}_t &= 2x_t + 2y_t + 6 \\ \dot{y}_t &= 2x_t - y_t + 12\end{aligned}$$

- We are also given initial values $x_0 = 1$ and $y_0 = 0$. We can rewrite this system in matrix form as

$$\begin{bmatrix} \dot{x}_t \\ \dot{y}_t \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

- Since we already know that the decomposition of A is then $A = C\Lambda C^{-1}$ where $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$, and $C^{-1} = \begin{bmatrix} 2/5 & 1/5 \\ 1/5 & -2/5 \end{bmatrix}$, if we define $Z_t = C^{-1}X_t \equiv \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix}$, the system of differential equations is now $\dot{Z}_t = \Lambda Z_t + C^{-1}b$, which is a simple system of univariate differential equations of the form

$$\begin{bmatrix} \dot{z}_{1t} \\ \dot{z}_{2t} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} + \begin{bmatrix} 24/5 \\ -18/5 \end{bmatrix}$$

- The solutions to the two univariate differential equations $\dot{z}_{1t} = 3z_{1t} + 24/5$ and $\dot{z}_{2t} = -2z_{2t} - 18/5$ can be easily calculated as $z_{1t} = De^{3t} - 8/5$, and $z_{2t} = Ee^{-2t} - 9/5$ using techniques we developed in the section on differential equations.
- We can then back out the solutions for X_t using the fact that $Z_t = C^{-1}X_t \Rightarrow X_t = CZ_t$. Therefore
$$X_t = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} De^{3t} - 8/5 \\ Ee^{-2t} - 9/5 \end{bmatrix} \equiv \begin{bmatrix} 2De^{3t} + Ee^{-2t} - 5 \\ De^{3t} - 2Ee^{-2t} + 2 \end{bmatrix}$$
- The initial conditions can be used to calculate $x_0 \equiv 1 = 2D + E - 5$ and $y_0 \equiv 0 = D - 2E + 2$, which simplifies to $D = 2$ and $E = 2$.
- The solutions to the system of differential equations is
$$\begin{aligned} x_t &= 4e^{3t} + 2e^{-2t} - 5 \\ y_t &= 2e^{3t} - 4e^{-2t} + 2 \end{aligned}$$
You can verify that this is indeed a solution to the system of differential equations we started with.

Dynamics

- The steady states of the system of differential equations $\dot{X}_t = AX_t + b$ can be found by setting $\dot{X}_t = 0$. We get $0 = AX^* + b \Rightarrow X^* = -A^{-1}b$.
- Consider the above example. Note that the eigenvalues of A are what appear in the exponential terms. So once again the stability of the system depends on the eigenvalues.
- If all the eigenvalues are < 0 then the system is said to be **stable**. If all the eigenvalues are ≥ 0 then the system is **unstable**. If at least one eigenvalue is < 0 the system is said to be **neutrally stable** or **saddlepath stable**.

V. HIGHER ORDER DIFFERENTIAL EQUATIONS

- As with difference equations, we can solve a higher-order differential equations by transforming it into a system of first order differential equations. Once again, I will illustrate using a second order differential equation, the method easily generalizes for an nth order differential equation.
- Consider a 2nd order differential equation of the form $\ddot{x}_t = \alpha_1\dot{x}_t + \alpha_2 x_t + \beta$. First, we will define an artificial variable $y_t = \dot{x}_t$. Then the single second order difference equation can be written as a system of first order difference equations of the form
$$\begin{aligned} \dot{y}_t &= \alpha_1 y_t + \alpha_2 x_t + \beta \\ \dot{x}_t &= y_t \end{aligned}$$
- Since this is a system of linear first order differential equations. we can then use the techniques from the last class to calculate a solution to the system, and by extension to the second order differential equation.

- This technique generalizes easily to an nth order differential equation. We first define $(n-1)$ artificial variables, each one conforming to a higher order derivative of the dependent variable. We then express the nth order differential equations as a system of n equations, the first being a first order differential equation in n variables and the remaining $(n-1)$ being identity equations.

- In order to find an exact solution we also need n starting values, not just a single initial starting value.

  Example

- Consider the following 2nd order difference equation: $\ddot{x}_t + \frac{1}{2}\dot{x}_t - \frac{1}{2}x_t = 5$ with initial conditions $x_0 = -8$ and $\dot{x}_0 = 4$.

- By defining an artificial variable $y_t = \dot{x}_t$, the second order differential equation can be written as a system of first order differential equations of the form

$$\begin{aligned} \dot{y}_t + \frac{1}{2}y_t - \frac{1}{2}x_t &= 5 \\ \dot{x}_t &= y_t \end{aligned}$$

- In matrix form, we have a system of equations of the form

$$\begin{bmatrix} \dot{y}_t \\ \dot{x}_t \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

- We have a system of first order linear differential equations of the form $\dot{X}_t = AX_t + b$, which can be solved by diagonalizing A. We will first find the eigenvalues and eigenvectors of A.

- The eigenvalues are the values of λ that solve $\begin{vmatrix} -1/2-\lambda & 1/2 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1/2\lambda - 1/2 = 0$ which are $\lambda = -1$ and $\lambda = 1/2$.

- The eigenvectors corresponding to $\lambda = -1$ satisfy $\begin{bmatrix} 1/2 & 1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, i.e. $\begin{bmatrix} a \\ -a \end{bmatrix}$.

- The eigenvectors corresponding to $\lambda = 1/2$ satisfy $\begin{bmatrix} -1 & 1/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, i.e. $\begin{bmatrix} a \\ 2a \end{bmatrix}$.

- Since the eigenvectors are linearly independent A is diagonalizable. The decomposition of A is then $A = C\Lambda C^{-1}$ where $\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 1/2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ and the inverse of C can be calculated as $C^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix}$.

- Define $Z_t = C^{-1}X_t \equiv \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix}$, the system of differential equations is now $\dot{Z}_t = \Lambda Z_t + C^{-1}b$, which is a simple system of univariate differential equations of the form

$$\begin{bmatrix} \dot{z}_{1t} \\ \dot{z}_{2t} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} + \begin{bmatrix} 10/3 \\ 5/3 \end{bmatrix}$$

- The solutions to the two univariate differential equations $\dot{z}_{1t} = -z_{1t} + 10/3$ and $\dot{z}_{2t} = z_{2t} + 5/3$ can be easily calculated as $z_{1t} = De^{-t} + 10/3$, and $z_{2t} = Ee^{\frac{1}{2}t} - 10/3$ using techniques we developed in the section on differential equations.

- We can then back out the solutions for X_t using the fact that $Z_t = C^{-1}X_t \Rightarrow X_t = CZ_t$.

$$X_t \equiv \begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} De^{-t} + 10/3 \\ Ee^{\frac{1}{2}t} - 10/3 \end{bmatrix} \equiv \begin{bmatrix} De^{-t} + Ee^{\frac{1}{2}t} \\ -De^{-1t} + 2Ee^{\frac{1}{2}t} - 10 \end{bmatrix}$$

- The initial conditions can be used to calculate $y_0 \equiv \dot{x}_0 = 4 \Rightarrow 4 = D + E$ and $x_0 \equiv -8 = -D + 2E - 10$, which simplifies to $D = 2$ and $E = 2$.

- The solutions to the system of differential equations is

$$\begin{aligned} y_t &= 2e^{-t} + 2e^{\frac{1}{2}t} \\ x_t &= -2e^{-1t} + 4e^{\frac{1}{2}t} - 10 \end{aligned}$$

  You can verify that this is indeed a solution to the system of differential equations we started with. You can also verify that $\dot{x} = y$ as it should be given the definition of the artificial variable y.