

LECTURE 19: Phase Diagram Analysis

I. INTRODUCTION

- Today's lecture discusses how to analyze the dynamics of a system of non-linear difference or differential equations using algebraic solutions and using phase diagrams. The phase diagrams in the two variable case are more complicated than in the single variable case because we have to keep track of the motion of two variables.
- However, this phase diagram technique will become extremely useful when we are dealing with a system of non-linear equations. It will also be useful when we do not want to explicitly solve a system of linear differential equations, but instead prefer to do comparative static analysis with the dynamics of the model.
- We will first work with a phase diagram for a system of linear differential equations and then draw the phase diagram for a system of linear difference equations.

II. PHASE DIAGRAMS FOR STABLE SYSTEMS

Example 1

- Consider the following system of first order differential equations

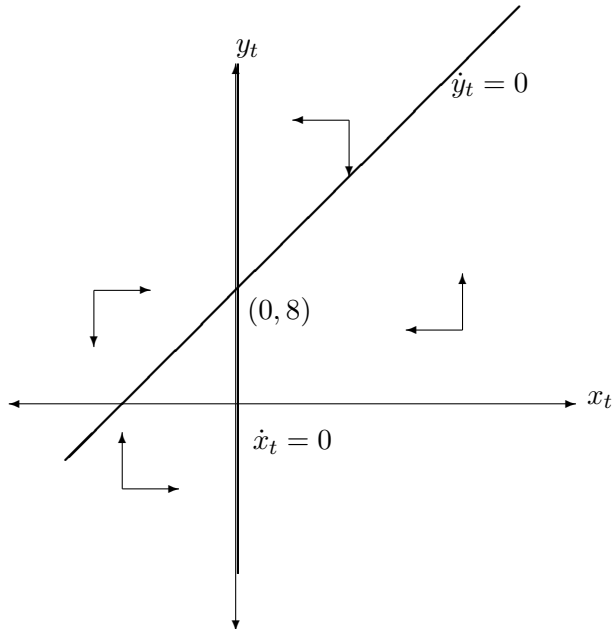
$$\begin{aligned}\dot{y}_t &= -4y_t + 2x_t + 32 \\ \dot{x}_t &= -2x_t\end{aligned}$$

- The solution to this system is

$$\begin{aligned}y_t &= Be^{-2t} + De^{-4t} + 8 \\ x_t &= Be^{-2t}\end{aligned}$$

- We can see that this is a stable system.
- Now look at the phase diagram. The locus of points corresponding to $\dot{x} = 0$ is easy, that is given by the equation $x_t = 0$, i.e. the y-axis. The locus of points corresponding to $\dot{y} = 0$ is $4y_t = 2x_t + 32$.
- The intersection of the two lines represents the set of points for which $\dot{x} = 0$ and $\dot{y} = 0$. The two lines intersect at $x = 0$ and $y = 8$.
- To the right of the $\dot{x} = 0$ line, $x > 0$ so $\dot{x} < 0$, i.e. x decreases over time (a left arrow in the diagram). Points to the left of the $\dot{x} = 0$ locus correspond to x increasing over time (a right arrow in the diagram).

- Similarly, to the right of the $\dot{y} = 0$ line, x is increasing and y is decreasing so $\dot{y}_t = -4y_t + 2x_t + 32 > 0$. In other words y is increasing over time (an up arrow in the diagram), while points to the left of the line are associated with decreasing y over time (a down arrow in the diagram).
- The phase diagram would look as follows.



- Notice that wherever we start, the dynamics of the system push us directly towards the steady state.

Example 2

- Sometimes, stability is more subtle. Consider the following system of first order differential equations

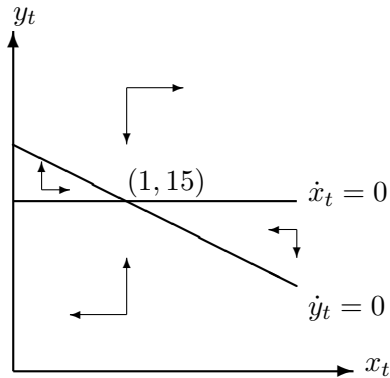
$$\begin{aligned}\dot{x}_t &= 3y_t - 45 \\ \dot{y}_t &= -x_t - 4y_t + 61\end{aligned}$$

- The solution is

$$\begin{aligned}x_t &= De^{-3t} - 3Ee^{-t} + 1 \\ y_t &= -De^{-3t} + Ee^{-t} + 15\end{aligned}$$

- The locus of points corresponding to $\dot{x} = 0$ is given by the equation $y_t = 15$. The locus of points corresponding to $\dot{y} = 0$ is $4y_t = 61 - x_t$.
- The two lines intersect at $x = 1$ and $y = 15$, which corresponds to a steady state of the system.

- Above the $\dot{x} = 0$, $y_t > 15$, so given the equation $\dot{x}_t = 3y_t - 45$ we can see that $\dot{x}_t > 0$, which we indicate with a right arrow. Below the $\dot{x} = 0$ line $\dot{x}_t < 0$, which we indicate with a left arrow
- Similarly, to the right of the $\dot{y} = 0$ line, x and y are both increasing so given $\dot{y}_t = -x_t - 4y_t + 61$, we can show that $\dot{y}_t < 0$, or y is decreasing, which we indicate with a down arrow. To the left of the $\dot{y} = 0$ line, $\dot{y}_t > 0$, or y is increasing, which we indicate with an up arrow.
- The phase diagram would look as follows.



- Note the difference in the dynamics. Although they never directly send you to steady state, the dynamics are always pushing you in towards the steady state and never letting you spiral outwards to ∞ or ∞ .

II. PHASE DIAGRAMS FOR UNSTABLE SYSTEMS

Example 1

- Consider the following system of first order differential equations

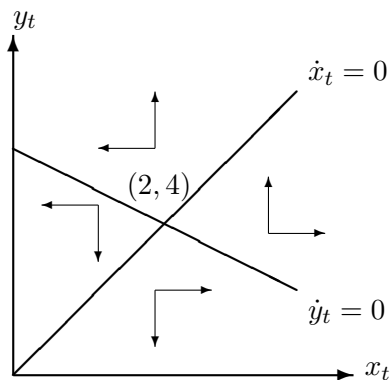
$$\begin{aligned}\dot{x}_t &= \frac{1}{2}x_t - \frac{1}{4}y_t \\ \dot{y}_t &= 13x_t + \frac{15}{2}y_t - 56\end{aligned}$$

- You can show that the solutions to the system of differential equations is

$$\begin{aligned}x_t &= -De^{7t} - Ee^t + 2 \\ y_t &= 26De^{7t} + 2Ee^t + 4\end{aligned}$$

- Once again instead of explicitly solving the system, we can draw a phase diagram to analyze the differential equation system. The locus of points corresponding to $\dot{x} = 0$ is given by the equation $y_t = 2x_t$.

- The locus of points corresponding to $\dot{y} = 0$ can be found by setting $\dot{y} = 0$ in the second equation to get $13x_t + 15/2y_t = 56$. We then draw these two lines on the phase diagram.
- The intersection of the two lines represents the set of points for which $\dot{x} = 0$ and $\dot{y} = 0$. i.e this would be the steady state of the model. The two lines intersect at $x = 2$ and $y = 4$, which corresponds to a steady state of the system.
- We can also plot the dynamics of the system using the phase diagram. To the right of the $\dot{x} = 0$ locus x is higher and y is lower than on the line. Given the equation $\dot{x}_t = 1/2x_t - 1/4y_t$ we can see that $\dot{x}_t > 0$ (a right arrow). To the left of the $\dot{x} = 0$ locus, $\dot{x}_t < 0$ (a left arrow).
- Similarly, we can show that to the right of the $\dot{y} = 0$ locus x is higher and y is higher than on the line. Given $\dot{y}_t = 13x_t + \frac{15}{2}y_t - 56$, it follows that $\dot{y}_t > 0$ (an up arrow in the diagram). To the left of the line $\dot{y}_t < 0$ (a down arrow in the diagram).
- From the arrows of the diagram, which indicate the dynamics of the model, we can see that the system is unstable, because wherever we start, the dynamics of the system push us away from the steady state towards ∞ or $-\infty$
- The phase diagram would look as follows.



Example 2

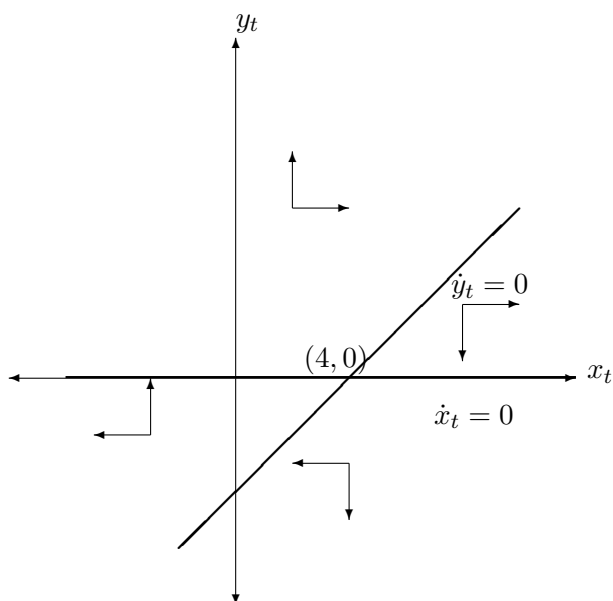
- Unstable systems can be far more subtle than the above. Consider the following system of first order differential equations

$$\begin{aligned}\dot{y}_t &= 6y_t - 8x_t + 32 \\ \dot{x}_t &= y_t\end{aligned}$$

- The solution to this system is

$$\begin{aligned}y_t &= 2Be^{2t} + 4De^{4t} \\ x_t &= Be^{2t} + De^{4t} + 4\end{aligned}$$

- We can see that this is an unstable system.
- Now look at the phase diagram. The locus of points corresponding to $\dot{x} = 0$ is easy, that is given by the equation $y_t = 0$, i.e. the x-axis. The locus of points corresponding to $\dot{y} = 0$ is $6y_t = 8x_t - 32$.
- The intersection of the two lines represents the set of points for which $\dot{x} = 0$ and $\dot{y} = 0$. The two lines intersect at $x = 4$ and $y = 0$.
- Above the $\dot{x} = 0$ line, $y > 0$ so $\dot{x} > 0$, i.e. x increases over time (a right arrow in the diagram). Points below the $\dot{x} = 0$ locus correspond to x decreasing over time (a left arrow in the diagram).
- Similarly, to the right of the $\dot{y} = 0$ line, x is increasing and y is decreasing. Given the equation $\dot{y}_t = 6y_t - 8x_t + 32 < 0$ we can see that y is decreasing over time (a down arrow in the diagram). Points to the left of the line are associated with increasing y over time (an up arrow in the diagram).
- The phase diagram would look as follows.



- Notice the counter-clockwise dynamics are similar to the second stable example above, except that in this case the dynamics would spiral in towards steady state instead of out. So how can you tell a stable spiral from an unstable one?
- A stable system will either have dynamics that indicate a direct path into the steady state or a more oblique counter-clockwise path into steady state.
- In both cases, the dynamics for a stable system should form a closed rectangle of the form:



III. PHASE DIAGRAMS FOR SADDLE-PATH STABLE SYSTEMS

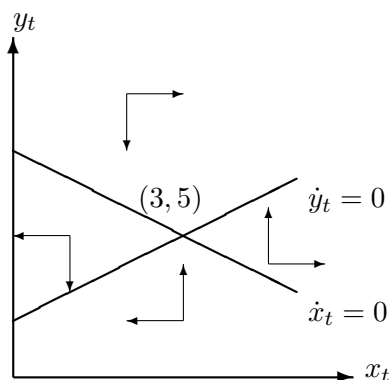
- Consider the following system of first order differential equations

$$\begin{aligned}\dot{x}_t &= \frac{5}{2}x_t + \frac{7}{2}y_t - 25 \\ \dot{y}_t &= \frac{1}{2}x_t - \frac{1}{2}y_t + 1\end{aligned}$$

- The solutions to this system of differential equations is

$$\begin{aligned}x_t &= 7De^{3t} - Ee^{-t} + 3 \\ y_t &= De^{3t} + Ee^{-t} + 5\end{aligned}$$

- From this system you can see that the system is saddle-path stable (one eigenvalue is < 0 but the other is > 1). It will only converge over time to a steady state of $x_t = 3$ and $y_t = 5$ for certain starting values (once that set $D=0$).
- Once again instead of explicitly solving the system, we can draw a phase diagram to analyze the differential equation system. The locus of points corresponding to $\dot{x} = 0$ is given by the equation $5x_t + 7y_t = 50$.
- The locus of points corresponding to $\dot{y} = 0$ can be found by setting $\dot{y} = 0$ in the second equation to get $x_t = y_t + 2$. We then draw these two locii, on the phase diagram.
- The intersection of the two lines represents the set of points for which $\dot{x} = 0$ and $\dot{y} = 0$. i.e this would be the steady state of the model. The two lines intersect at $x = 3$ and $y = 5$, which corresponds to a steady state of the system.
- We can also plot the dynamics of the system using the phase diagram. To the right of the $\dot{x}_t = 0$ line x is higher and y is higher than on the line. Given the equation $\dot{x}_t = 5/2x_t + 7/2y_t - 25$ we can see that $\dot{x}_t > 0$. In other words points to the right of the $\dot{x} = 0$ locus correspond to x increasing over time (a right arrow in the diagram). Similarly, to the left of the $\dot{x}_t = 0$ line, $\dot{x}_t < 0$, x decreases over time (a left arrow in the diagram).
- The phase diagram would look as follows.



- From the arrows of the diagram, which indicate the dynamics of the model, we can see that the system is unstable in some regions, i.e. the dynamics of the system push us away from the steady state towards ∞ or $-\infty$. But in certain other quadrants, the dynamics lead us towards the steady state of $(3, 5)$. The system is therefore said to be saddle-path stable.

Analytically of the Stable Arm

- Let's focus on the analytical solution to develop some understanding of the shape of the stable arm.
- When we evaluate the system at $t = 0$, we get

$$\begin{aligned}x_0 &= 7D - E + 3 \\y_0 &= D + E + 5\end{aligned}$$

- We can solve for the values of D and E for any given set of starting values. By adding the two equations together we eliminate E and get

$$\begin{aligned}x_0 + y_0 &= 8D + 8 \\D &= \frac{(x_0 + y_0)}{8} - 1\end{aligned}$$

- We can then solve for E as

$$E = \frac{(7y_0 - x_0)}{8} - 4$$

- Along the stable arm, the system converges to a steady state. So we need $D = 0$ otherwise the system will diverge as t gets larger. This will only be true when $x_0 + y_0 = 8$
- The stable arm is the set of points that satisfies $x + y = 8$. So any starting value on the stable arm must satisfy $x_0 + y_0 = 8$. Any starting point on this line is consistent with stability of the system.
- Given that $D = 0$, the system reduces to

$$\begin{aligned}x_t &= -Ee^{-t} + 3 \\y_t &= Ee^{-t} + 5\end{aligned}$$

- So if we had started at $x_0 = 0$ and $y_0 = 8$ the solution would be

$$\begin{aligned}x_t &= (-3)e^{-t} + 3 \\y_t &= 3e^{-t} + 5\end{aligned}$$

- Alternatively, if we had started at $x_0 = 2$ and $y_0 = 6$ the solution would be

$$\begin{aligned}x_t &= -e^{-t} + 3 \\y_t &= e^{-t} + 5\end{aligned}$$

- In either case, we would converge to the steady state of $(3, 5)$