Fall Semester '05-'06
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LECTURE 20: MARKOV CHAINS

I. INTRODUCTION

- Today's lecture focuses on an important class of economic models that are built on a system of first order difference equations.
- These models, known as 'Markov models' or 'Markov chains' relate economic variables today to their values in the previous period alone. These models are very tractable to work with since only last period's values affect next periods outcome (i.e. we don't need to carry around a long history).
- These models are sometimes known as 'memoryless' models because of the feature that only the immediate history (and not all of past history) matter for solving the model.
- We will illustrate using two examples: a simple labor market model and a model that describes population growth.

II. PROPERTIES OF MARKOV CHAINS

- A **Markov chain** is a system of first order difference equations that relates transitions between different states of the world over time. For example we could use a Markov Chain to describe how individuals transit between the states "unemployed" and "employed" between two periods in time.
- The relationship between the state of the world in time $t-1$ and the state of the world in time t is given by a matrix of coefficients known as a transition matrix.
- A typical Markov chain model is of the form $\Pi_t = P\Pi_{t-1}$ where Π_t is an $n \times 1$ vector of probabilities describing the state being modeled at time t, and P is an $n \times n$ square matrix that describes the transition probabilities of moving from a given state to any of the other states.
- To put it another way, the ith element of Π denotes the probability that state of the world i has been realized at time t. The i,jth entry of P, P_{ij}, denotes the probability that an item that is in state j at time $t-1$ will be in state i at time t.
- Since P is a matrix of transition probabilities it has to follow certain rules. None of the entries in P can be negative since a transition probability always has to be non-negative. None of the entries can be > 1 either since probabilities have an upper bound of 1.
- Furthermore, since $\sum\limits_{i=1}^{n} P_{ij}$ denotes the probability that an item that is in state j at time $t-1$ will be in any one of the other n states at time t, we need $\sum\limits_{i=1}^{n} P_{ij} = 1$, i.e. we need all the columns of P to add up to 1.

- Given a Markov chain of the form $\Pi_t = P\Pi_{t-1}$, through repeated iteration, we know that the solution is of the form $\Pi_t = P^t\Pi_0$ where Π_0 is the initial distribution of probabilities.
- If P is diagonalizable then the decomposition of P is then $P = C\Lambda C^{-1}$. In a prior lecture, we also showed that $P^t = C\Lambda^t C^{-1}$. So the solution to the Markov chain is relatively easy to calculate as $\Pi_t = C\Lambda^t C^{-1}\Pi_0$.
- Markov Chains will always have at least one eigenvalue whose value is equal to 1. Furthermore, the remaining eigenvalues will be ≤ 1 in absolute value.

Example 1: Unemployment Model

- In any given period an unemployed individual can find a job with probability 15% while an employed person can lose their job with probability 1%. These are the only two states of the world (i.e. being out of the labor force is not an option).
- The economy starts out with 25% unemployed and 75% employed. In other words

$$\Pi_0 = \begin{bmatrix} U_0 \\ E_0 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}$$

- What we have is a system of first order linear difference equations of the form

$$\begin{bmatrix} U_t \\ E_t \end{bmatrix} = \begin{bmatrix} 0.85 & 0.01 \\ 0.15 & 0.99 \end{bmatrix} \begin{bmatrix} U_{t-1} \\ E_{t-1} \end{bmatrix}$$

- The matrix of transition probabilities is given by

$$P = \begin{bmatrix} 0.85 & 0.01 \\ 0.15 & 0.99 \end{bmatrix}$$

- Note that the conditions for the transition matrix are satisfied. Using this information we can calculate a time path for the number of unemployed and the number of employed in the economy. We can also calculate a time path for the unemployment rate in the economy.
- We need to first find the eigenvalues and eigenvectors of P. The eigenvectors are the values of λ that solve $\left| \begin{matrix} 0.85-\lambda & 0.01 \\ 0.15 & 0.99-\lambda \end{matrix} \right] = 0 \Rightarrow \lambda^2 - 1.84\lambda + 0.84 = 0$ which are $\lambda = 1$ and $\lambda = 0.84$. Since the eigenvalues are distinct, we know that the matrix is diagonalizable.
- The eigenvectors corresponding to $\lambda = 1$ are of the form

$$\begin{bmatrix} -0.15 & 0.01 \\ 0.15 & -0.01 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 15 \end{bmatrix}$$

- The eigenvectors corresponding to $\lambda = 0.84$ are of the form

$$\begin{bmatrix} 0.01 & 0.01 \\ 0.15 & 0.15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

- The decomposition of P is then $P = C\Lambda C^{-1}$ where $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0.84 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 15 & -1 \end{bmatrix}$.
- The system of difference equations is $\Pi_t = C\Lambda C^{-1}\Pi_{t-1}$. Defining a new variable $Z_t = C^{-1}\Pi_t$ we can write the system as $Z_t = \Lambda Z_{t-1}$ or

$$\begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.84 \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix}$$

- The solution to this is very easy to find, $z_{1t} = B$ and $z_{2t} = D(0.84)^t$ for arbitrary constants B, D. Since $\Pi_t = CZ_t$ we can find that

$$\begin{bmatrix} U_t \\ E_t \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 15 & -1 \end{bmatrix} \begin{bmatrix} B \\ D(0.84)^t \end{bmatrix} = \begin{bmatrix} B + D(0.84)^t \\ 15B - D(0.84)^t \end{bmatrix}$$

- We can eliminate the arbitrary constants by plugging in $t = 0$ to get $U_0 = B + D$, $E_0 = 15B - D$, which imply that $B = \frac{1}{16}(U_0 + E_0)$ and $D = \frac{1}{16}(15U_0 - E_0)$.
- Now we have the complete solution for the probability of being unemployed or employed at any given time as

$$\begin{aligned} U_t &= \frac{1}{16}(U_0 + E_0) + (0.84)^t \frac{1}{16}(15U_0 - E_0) \\ E_t &= \frac{15}{16}(U_0 + E_0) - (0.84)^t \frac{1}{16}(15U_0 - E_0) \end{aligned}$$

- Given our initial probabilities of 25% unemployed and 75% employed we get

$$\begin{aligned} U_t &= \left(\frac{1}{16}\right) + (0.84)^t \left(\frac{3}{16}\right) \\ E_t &= \left(\frac{15}{16}\right) - (0.84)^t \left(\frac{3}{16}\right) \end{aligned}$$

- The unemployment rate can can be calculated as

$$u \equiv \frac{U}{E+U} = \frac{\left(\frac{1}{16}\right) + (0.84)^t \left(\frac{3}{16}\right)}{\left(\frac{15}{16}\right) - (0.84)^t \left(\frac{3}{16}\right) + \left(\frac{1}{16}\right) + (0.84)^t \left(\frac{3}{16}\right)} = \left(\frac{1}{16}\right) + (0.84)^t \left(\frac{3}{16}\right)$$

- The unemployment rate in the long run (the natural rate of unemployment) is 1/16 or 6.25%

Example 2: Business Cycle Forecasts

- We can use Markov Chains to model the likelihood of a recession in the future. Suppose there are two states that an economy can be in Recession (R) and boom (B).
- Suppose that there is a r chance that an economy will go into recession and a b chance that an economy in recession will recover.
- We can represent this using a Markov chain as

$$\begin{bmatrix} R_t \\ B_t \end{bmatrix} = \begin{bmatrix} (1-b) & r \\ b & (1-r) \end{bmatrix} \begin{bmatrix} R_{t-1} \\ B_{t-1} \end{bmatrix}$$

- The transitions of the state of the economy from one period to the next is given by $\Pi_t = P\Pi_{t-1}$. If we diagonalize P as $C\lambda C^-1$ we can express the value of Π_t as $\Pi_t = C\Lambda C^{-1}\Pi_{t-1}$

- The eigenvalues are the values for which

$$\begin{vmatrix} 1-b-\lambda & r \\ b & 1-r-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - (2-b-r)\lambda + (1-b-r) = 0$$

- The roots are $\lambda = 1$, $\lambda = 1-b-r$.

- The eigenvector corresponding to $\lambda = 1$ is of the form $\begin{bmatrix} r \\ b \end{bmatrix}$. The eigenvector corresponding to $\lambda = 1-b-r$ is of the form $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

- The C matrix can therefore be written as

$$\begin{bmatrix} r & 1 \\ b & -1 \end{bmatrix}$$

- Defining a new variable $Z_t = C^{-1}\Pi_t$ we can write the system as $Z_t = \Lambda Z_{t-1}$ or

$$\begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1-b-r \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \end{bmatrix}$$

- The solution to this is very easy to find, $z_{1t} = E$ and $z_{2t} = F(1-b-r)^t$ for arbitrary constants E and F. Since $\Pi_t = CZ_t$ we can find that

$$\begin{bmatrix} R_t \\ B_t \end{bmatrix} = \begin{bmatrix} rE + F(1-b-r)^t \\ bE - F(1-b-r)^t \end{bmatrix}$$

- We can eliminate the arbitrary constants E and F by plugging in $t = 0$ to get $R_0 = rE + F$, $B_0 = bE - F$ which imply that $E = \frac{R_0+B_0}{r+b}$, $F = \frac{bR_0 - rB_0}{r+b}$.

- Now we have the complete solution for the state of the economy at any given time as

$$\begin{aligned} R_t &= \left(\frac{r}{r+b}\right)(R_0 + B_0) + \left(\frac{bR_0 - rB_0}{r+b}\right)(1-b-r)^t \\ B_t &= \left(\frac{b}{r+b}\right)(R_0 + B_0) - \left(\frac{bR_0 - rB_0}{r+b}\right)(1-b-r)^t \end{aligned}$$

- If the economy starts out in recession $R_0 = 1$, the likelihood that it is in recession or boom at time t are given by (respectively)

$$\begin{aligned} R_t &= \left(\frac{r}{r+b}\right) + \left(\frac{b}{r+b}\right)(1-b-r)^t \\ B_t &= \left(\frac{b}{r+b}\right) - \left(\frac{b}{r+b}\right)(1-b-r)^t \end{aligned}$$

- If the economy starts out in boom $B_0 = 1$, the likelihood that it is in recession or boom at time t are given by (respectively)

$$
\begin{aligned}
R_t &= \left(\frac{r}{r+b}\right) - \left(\frac{r}{r+b}\right)(1-b-r)^t \\
B_t &= \left(\frac{b}{r+b}\right) + \left(\frac{r}{r+b}\right)(1-b-r)^t
\end{aligned}
$$

- In the long run we can see that the economy is in recession with a constant probability of $\frac{r}{r+b}$ and in a boom with a constant probability of $\frac{b}{r+b}$.

Example 3: Models of Disease

- We can use Markov Chains to model the spread of a disease. Suppose there are three states of the world sick (S), healthy (H) and dead (D).
- Consider a disease in which 20% of the sick people die, 15% of the healthy people get infected. Also suppose that 5% of sick people recover completely to good health. Finally we assume that healthy people don't die.
- We can represent this using a Markov chain as

$$
\begin{bmatrix} H_t \\ S_t \\ D_t \end{bmatrix} = \begin{bmatrix} 0.85 & 0.05 & 0 \\ 0.15 & 0.75 & 0 \\ 0.00 & 0.20 & 1 \end{bmatrix} \begin{bmatrix} H_{t-1} \\ S_{t-1} \\ D_{t-1} \end{bmatrix}
$$

- Suppose that the population starts out with everyone being healthy.
- The probability at time t of being sick, healthy or dead Π_t can then be expressed as $\Pi_t = P\Pi_{t-1}$. If we diagonalize P as $C\lambda C^-1$ we can express the value of Π_t as $\Pi_t = C\Lambda C^{-1}\Pi_{t-1}$
- The eigenvalues are the values for which

$$
\begin{vmatrix} 0.85-\lambda & 0.05 & 0 \\ 0.15 & 0.75-\lambda & 0 \\ 0.00 & 0.20 & 1-\lambda \end{vmatrix} = 0
$$

- The resulting cubic equation is $(1-\lambda)[\lambda^2 - 1.60\lambda + (0.6375) - (0.0075)] \equiv (1-\lambda)[\lambda^2 - 1.60\lambda + 0.63] = 0$ The roots are $\lambda = 1$, $\lambda = 0.9$ and $\lambda = 0.7$.
- The eigenvector corresponding to $\lambda = 1$ is of the form $\begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}$. The eigenvector corresponding to $\lambda = 0.9$ is of the form $\begin{bmatrix} -a \\ -a \\ 2a \end{bmatrix}$ and the eigenvector corresponding to $\lambda = 0.7$ is of the form $\begin{bmatrix} a \\ -3a \\ 2a \end{bmatrix}$

- The C matrix can therefore be written as

$$\begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

- Defining a new variable $Z_t = C^{-1}\Pi_t$ we can write the system as $Z_t = \Lambda Z_{t-1}$ or

$$\begin{bmatrix} z_{1t} \\ z_{2t} \\ z_{3t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \\ z_{3t-1} \end{bmatrix}$$

- The solution to this is very easy to find, $z_{1t} = E$ and $z_{2t} = F(0.9)^t$ and $z_{3t} = G(0.7)^t$ for arbitrary constants E, F and G. Since $\Pi_t = CZ_t$ we can find that

$$\begin{bmatrix} H_t \\ S_t \\ D_t \end{bmatrix} = \begin{bmatrix} -F(0.9)^t + G(0.7)^t \\ -F(0.9)^t - 3G(0.7)^t \\ E + 2F(0.9)^t + 2G(0.7)^t \end{bmatrix}$$

- We can eliminate the arbitrary constants E, F and G by plugging in $t = 0$ to get $H_0 \equiv 1 = G - F$, $S_0 \equiv 0 = -F - 3G$, and $D_0 \equiv 0 = E + 2F + 2G$ which imply that $G = \frac{1}{4}$, $F = -\frac{3}{4}$ and $E = 1$.

- Now we have the complete solution for the probability of being healthy, sick or dead as

$$\begin{aligned} H_t &= \frac{3}{4}(0.9)^t + \frac{1}{4}(0.7)^t \\ S_t &= \frac{3}{4}(0.9)^t - \frac{3}{4}(0.7)^t \\ D_t &= 1 - \frac{3}{2}(0.9)^t + \frac{1}{2}(0.7)^t \end{aligned}$$

- In the long run we can see that both H_t and S_t approach zero, while D_t approaches 1. This of course proves one of the more famous claims in economics. Do you know what it is?

Example 4: Inventory Allocation Models

- A rental car company has 3 locations: BOSTON (B), LOS ANGELES (L) and CHICAGO (C).

- In any given month 50% of the rental cars in Chicago move to LA and the remaining 50% move to Boston. In the meantime 50% of the cars that were in Boston move to Chicago while the remainder stay in Boston. Similarly, 50% of the cars that were in LA. move to Chicago while the remainder stay in LA. Suppose we begin with 50% of the cars in Chicago and 25% each in Boston and LA.

- We can represent this using a Markov chain as

$$\begin{bmatrix} B_t \\ C_t \\ L_t \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} B_{t-1} \\ C_{t-1} \\ L_{t-1} \end{bmatrix}$$

- The probability that a rental cars at time t is in each city can then be expressed as $\Pi_t = P\Pi_{t-1}$. If we diagonalize P as $C\lambda C^{-}1$ we can express the value of Π as $\Pi_t = C\Lambda C^{-1}\Pi_{t-1}$

- The eigenvalues are the values for which

$$\begin{vmatrix} 0.5-\lambda & 0.5 & 0 \\ 0.5 & -\lambda & 0.5 \\ 0 & 0.5 & 0.5-\lambda \end{vmatrix} = 0$$

- The resulting cubic equation is $-\lambda(0.5-\lambda)^2 - 2(0.5^2)(0.5-\lambda) = 0$. This simplifies to $(0.5-\lambda)[-\lambda(0.5-\lambda)-0.5] = (0.5-\lambda)[\lambda^2 - 0.5\lambda - 0.5] = 0$, or

$$(0.5-\lambda)(\lambda-1)(\lambda+0.5) = 0$$

- The roots are $\lambda = 1$, $\lambda = 0.5$ and $\lambda = -0.5$.

- The eigenvector corresponding to $\lambda = 1$ is of the form $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The eigenvector corresponding to $\lambda = 0.5$ is of the form $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and the eigenvector corresponding to $\lambda = -0.5$ is of the form $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

- The C matrix can therefore be written as

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

- Defining a new variable $Z_t = C^{-1}\Pi_t$ we can write the system as $Z_t = \Lambda Z_{t-1}$ or

$$\begin{bmatrix} z_{1t} \\ z_{2t} \\ z_{3t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} z_{1t-1} \\ z_{2t-1} \\ z_{3t-1} \end{bmatrix}$$

- The solution to this is very easy to find, $z_{1t} = E$ and $z_{2t} = F(0.5)^t$ and $z_{3t} = G(-0.5)^t$ for arbitrary constants E, F and G. Since $\Pi_t = CZ_t$ we can find that

$$\begin{bmatrix} B_t \\ C_t \\ L_t \end{bmatrix} = \begin{bmatrix} E - F(0.5)^t + G(-0.5)^t \\ E - 2G(-0.5)^t \\ E + F(0.5)^t + G(-0.5)^t \end{bmatrix}$$

- We can eliminate the arbitrary constants E, F and G by plugging in $t = 0$ to get $B_0 \equiv .25 = E - F + G$, $C_0 \equiv .5 = E - 2G$, and $L_0 \equiv .25 = E + F + G$ which imply that $E = \frac{1}{3}$, $F = 0$ and $G = -\frac{1}{12}$.

- Now we have the complete solution for the probability distribution of cars in each city at any given time as

$$\begin{aligned} B_t &= \frac{1}{3} - \frac{1}{12}(-0.5)^t \\ C_t &= \frac{1}{3} - \frac{1}{6}(-0.5)^t \\ L_t &= \frac{1}{3} + \frac{1}{12}(-0.5)^t \end{aligned}$$

- In the long run we can see that the inventory converges to an equal distribution of cars in each city.