Fall Semester '05-'06
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Lecture 25: Models with Hamiltonians

I. INTRODUCTION

- In the last lectures we looked at techniques for solving dynamic optimization problems in continuous time. Using a Hamiltonian function, we derived a FOC and a co-state equation that were the continuous time analog to the FOC and the envelope condition from the Bellman formulation.
- We also combined these equations to form the Euler equation in continuous time. This was more complicated than in the discrete case by the fact that we could not pose it as a reallocation over two discrete periods. Instead, what the Euler equation is telling us is a story about the growth path of the choice variable.
- In today's class, we will explore some more applications of the Hamiltonian to make sure that you are comfortable with the concepts and the solution techniques.

II. MODELS WITH HAMILTONIAN EQUATIONS

Profit Maximization

- Consider the following multi-period firm profit maximization decision in continuous time

$$\max_{I_t, L_t} \int_{t=0}^{T} e^{-rt} \left[F(K_t, L_t) - w_t L_t - \rho_t I_t\right] dt$$

subject to the constraints $\dot{K}_t = -\delta K_t + I_t$ and K_0, K_T are given.

- The Hamiltonian is

$$H_t = \Big[F(K_t, L_t) - w_t L_t - \rho_t I_t + \mu_t(\dot{K}_t)\Big] \text{ where } \dot{K}_t = -\delta K_t + I_t$$

- The FOCs of this model are

$$\begin{aligned}
\frac{\partial H_t}{\partial L_t} = 0 \quad &\Rightarrow \quad F_L(K_t, L_t) - w_t = 0 \\
\frac{\partial H_t}{\partial I_t} = 0 \quad &\Rightarrow \quad -\rho_t + \mu_t = 0 \\
\frac{\partial H_t}{\partial K_t} = r\mu_t - \dot{\mu}_t \quad &\Rightarrow \quad F_K(K_t, L_t) - \delta\mu_t = r\mu_t - \dot{\mu}_t
\end{aligned}$$

- The first equation says that the marginal product of labor is set equal to its marginal cost. Since labor hiring is not dynamic we get the same condition as in the static case.

- Combining the last two equations results in the following equation

$$F_K(K_t, L_t) = (r+\delta)\rho_t - \dot{\rho}_t$$

- This is the Euler equation for this problem: it states in a different way the core intuition from the discrete case: that along the optimal path of investment, there can be no gains from rearranging the path of investment.
- What is the interpretation here? Well, the flow cost of adding an additional investment unit is $(r+\delta)\rho_t$ i.e. we are paying a constant stream of costs that are basically the depreciation rate plus the opportunity cost in terms of foregone real interest we could have earned on the money. The flow gain from investing is $F_K(K_t, L_t) + \dot{\rho}_t$, the marginal product of capital plus any change in price of capital (if the price of capital rises over time, then that increases the benefit of having already invested).
- This is identical to the discrete case where we had the Euler equation

$$\rho_t = \left(\frac{1}{1+r}\right)\frac{\partial F}{\partial K_{t+1}} + \left(\frac{1}{1+r}\right)(1-\delta)\rho_{t+1}$$

  This can be rearranged as $(1+r)\rho_t = \frac{\partial F}{\partial K_{t+1}} + (1-\delta)\rho_{t+1}$. Doing some algebra, we get

$$r\rho_t + \delta\rho_{t+1} - (\rho_{t+1} - \rho_t) = \frac{\partial F}{\partial K_{t+1}}$$

  which is identical to the above when we don't have discrete jumps in time.

Infinite Horizon Problems

- How do we modify the Hamiltonian technique when were are facing an infinite horizon problem, i.e. when the objective function is $\max_{x_1,\cdots,x_\infty} \sum_{t=1}^{\infty} e^{-\rho t} f(x_t, A_t)$ instead of $\max_{x_1,\cdots,x_T} \sum_{t=1}^{T} e^{-\rho t} f(x_t, A_t)$
- As in the discrete time case, we do not have to change anything in the solution technique that we use. We would continue to use a FOC and a co-state equation, and then combine them to obtain an Euler equation in continuous time.
- In a finite horizon case, we combine this with the initial condition $A_0 = \underline{A}$ and the terminal condition $A_T = \overline{A}$ to solve for the timepaths of the choice variable and the state variable. In the infinite horizon problem, we don't have a well-defined terminal condition, so we use a transversality condition of the form:
- The condition that is typically used in an infinite horizon problem is called a "transversality condition". The definition of the transversality condition is

$$\lim_{t\to\infty} A_t\mu_t = 0$$

- What this condition is saying is the following: as we go further and further into the future, we must either have zero units of the state variable left ($A_t = 0$) or if we have non-zero units of the state variable left, then those extra units must not contribute anything to maximized utility $\mu_t A_t = 0$.

- In the example above, w the transversality condition collapses down to

$$\lim_{t\to\infty} K_t \rho_t = 0$$

in other words, we either leave behind no capital, or if we leave capital behind, it is because capital does not fetch anything on the open market or else we could have increased profits by selling off the capital.

The Ramsey Model

- One of the most famous macroeconomic models is the Ramsey/Cass/Koopman's model of an economy. The following is a simplified version of that model describing the behavior of a consumer/producer in the economy.
- The individual faces the following maximization decision

$$\max_{C_t} \int_0^\infty \left[e^{-\rho t} u(C_t)\right] dt \text{ subject to } \dot{K}_t = F(K_t) - C_t - \delta K_t$$

- C_t is consumption, K_t is the capital stock and $0 < \delta < 1$ is the rate of depreciation in capital.
- The utility function is

$$u(C_t) = \frac{\left(C_t^{1-\frac{1}{\sigma}}\right) - 1}{1 - \frac{1}{\sigma}}$$

where $\sigma > 0$ and the production function is $F(K_t) = K_t^\alpha$ where $0 < \alpha < 1$

- We can define the Hamiltonian as

$$H_t = U(C_t) + \mu_t \dot{K}_t \equiv U(C_t) + \mu_t[F(K_t) - C_t - \delta K_t]$$

- The initial level of capital K_0 is given, and we also assume that the transversality condition holds, so that

$$\lim_{t\to\infty} K_t \mu_t = 0$$

- The first order condition and the co-state equation are

$$\frac{\partial H}{\partial C_t} = 0 \qquad \Rightarrow U'(C_t) - \mu_t = 0$$

$$\frac{\partial H}{\partial K_t} = \rho\mu_t - \dot{\mu}_t \qquad \Rightarrow [F'(K_t) - \delta]\mu_t = \rho\mu_t - \dot{\mu}_t$$

- Combining the first two equations gives us the Euler equation for consumption in continuous time.

$$[F'(K_t) - \delta]U'(C_t) = \rho U'(C_t) - U''(C_t)\dot{C}_t$$

- This simplifies to the following Euler equation

$$U''(C_t)\dot{C}_t = (\rho + \delta - F'(K_t))U'(C_t)$$

- Given the utility function $U(C_t) = 2\sqrt{C_t}$ and the production function $F(K_t)K_t^\alpha$ we have $F'(K_t) = K_t^\alpha$, $U'(C_t) = \frac{1}{\sqrt{C_t}}$ and $U''(C_t) = \frac{-1}{2C_t^{3/2}}$.

- We can rearrange the above equation so that it reads $\frac{-\dot{C}}{C_t\sqrt{C_t}} = 2(\rho + \delta - \alpha K_t^{\alpha-1})\frac{1}{\sqrt{C_t}} \Rightarrow \frac{\dot{C}}{C_t} = 2(\alpha K_t^{\alpha-1} - \delta - \rho)$

- The growth rate of consumption is related to the difference between the the rate of return on capital $(F'(K) - \delta)$ and the discount rate ρ. If the rate of return on capital and the rate at which we discount the future are identical, i.e. $F'(K) - \delta = \rho$ then consumption will remain constant over time (analogous to the $\beta(1 + r) = 1$ case in discrete time optimization).

- If on the other hand, the rate of return on capital exceeds the rate at which we discount the future, i.e. $F'(K) - \delta > \rho$ then consumption will be rising over time (analogous to the $\beta(1 + r) > 1$ case in discrete time optimization).

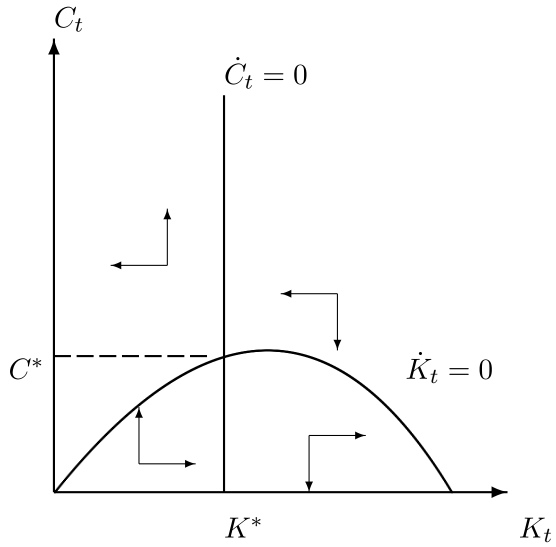
- Finally, if the rate of return on capital is less than the rate at which we discount the future, i.e. $F'(K) - \delta < \rho$ then consumption will be falling over time (analogous to the $\beta(1 + r) < 1$ case in discrete time optimization).

- Note that the Euler equation $\frac{\dot{C}}{C} = 2(\alpha K^{\alpha-1} - \delta - \rho)$ along with the flow budget constraint $\dot{K}_t = K_t^\alpha - \delta K_t - C_t$ and the initial condition $K_0 = \underline{K}$ and transversality condition $\lim_{t\to\infty} K_t U'(C_t)$ constitute a system of non-linear differential equations that make up the solution to the problem.

- We can do further analysis by drawing a phase diagram on which we indicate the two collections of points corresponding to $\dot{C} = 0$ and $\dot{K} = 0$.

- The set of points corresponding to the $\dot{C} = 0$ locus is $K_t = \left(\frac{\alpha}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}$ To the right of this line $\dot{C} < 0$ $(C \downarrow)$ and to the left of this line $\dot{C} > 0$ $(C \uparrow)$

- The set of points corresponding to the $\dot{K} = 0$ locus is $C_t = K_t^\alpha - \delta K_t$. This is a concave line with two horizontal intercepts at $K = 0$ and $K = \left(\frac{1}{\delta}\right)^{\frac{1}{1-\alpha}}$. Above this line $\dot{K} < 0$ $(K \leftarrow)$ and below this line $\dot{K} > 0$ $(K \rightarrow)$. The phase diagram is below.



- We can show that the system is saddle-path stable and that the steady state of the model is at

$$K^* = \left(\frac{\alpha}{\delta + \rho}\right)^{\frac{1}{1-\alpha}}$$

$$C^* = \left(\frac{\alpha}{\delta + \rho}\right)^{\frac{\alpha}{1-\alpha}} - \delta \left(\frac{\alpha}{\delta + \rho}\right)^{\frac{1}{1-\alpha}}$$