1 Introduction

Terminology:

- Let $A \in \mathbb{Q}^{u \times v}$. We refer to $A$ as being $CC(m)$ if $A$ satisfies the columns condition with a partition consistent of $m$ classes.

- By Rado’s theorem we have that $A$ is $CC(m)$ for some $m \in \mathbb{N}$ if and only if $A$ is partition regular. We will use the abbreviation $PR$ for partition regular and use it interchangeably with the statement “$A$ is $CC(m)$ for some $m \in \mathbb{N}$.”

- Let $A \in \mathbb{Q}^{u \times v}$. An index $k$ is a null index if for every vector $\mathbf{x} = [x_i] \in \ker A$, $x_k = 0$.

Observation 1.1.

1. $A \in CC(1)$ if and only if $A\mathbf{1} = \mathbf{0}$, where $\mathbf{1} = [1, \ldots, 1]^T$.

2. If $A$ is $CC(m)$, then $\text{rank}(A) \leq v - m$, since in this case $A$ has at least $m$ dependent columns.

3. If $\mathbf{x} \in \ker(A)$ such that no coordinate of $\mathbf{x}$ is allowed to be 0, and $D = \text{diag}(x_1, \ldots, x_v)$, then the matrices $D^{-1}AD$ and $AD$ are $CC(1)$ and therefore PR.

Theorem 1.2. If $A$ is a $u \times v$ matrix such that $A$ has a null index, then $A$ is not $PR$. 
2 Incidence matrices of oriented graphs

Adjacency matrices are not PR since they are nonnegative and nonzero matrices. Oriented vertex-edge incidence matrices can be PR and we look here at questions that arise naturally in this context.

Notation:
Let $\vec{G} = (V, E)$ be an oriented graph. Then $D_{\vec{G}}$ denotes its vertex-edge incidence matrix.

Observation 2.1. While an arbitrary matrix $A$ is $CC(1)$ if and only if $1 \in \ker(A)$, for any oriented graph $\vec{G}$, the vector $1 = [1, \ldots, 1]^T$ is in the left null space of $D_{\vec{G}}$.

Observation 2.2. Let $\vec{G}$ be an oriented graph and $\vec{D}_{\vec{G}}$ its vertex-edge incidence matrix. If $\vec{G}$ has either a source or a sink then $\vec{D}_{\vec{G}}$ is not PR.

Observation 2.3. For any $\vec{G}$, the matrix $D_{\vec{G}}$ has net column weight of 0. Note that the row weight is variable.

Theorem 2.4. Let $G$ be a connected graph. The following are equivalent:
1. $K_e(G) \geq 2$, i.e., $G$ has no bridge,
2. $G$ is the union of its cycles,
3. $G$ can be oriented so that the corresponding vertex-edge incidence matrix is PR.

Proof. The following recursive algorithm produces the desired partition: Pick an unoriented cycle and orient it cyclically. Let the corresponding columns of the vertex-edge incidence matrix be the first cell of our partition $I_1$. Repeat this process until all edges are in some class $I_k$. \hfill $\square$

Theorem 2.5. For an oriented graph $\vec{G}$, the matrix $\vec{D}_{\vec{G}}$ is PR if and only if $\vec{G}$ is strongly connected.

Corollary 2.6. For any $D_{\vec{G}}$, where $\vec{G}$ is strongly connected,

$$\text{rank } D_{\vec{G}} \leq |G| - 1.$$ 

Observation 2.7. If $C_n$ is an oriented cycle on $n$ vertices then $D_{C_n}$ is $CC(1)$ and $\text{rank } D_{C_n} = n - 1$.

Theorem 2.8. Let $G$ be any graph which contains a Hamiltonian cycle. Then $G$ can be oriented in such a way that $D_{\vec{G}}$ is $CC(2)$ and therefore PR.

Theorem 2.9. If $\vec{D}_{\vec{G}}^1$ and $\vec{D}_{\vec{G}}^2$ are two distinct orientated vertex-edge incidence matrices associated with a graph $G$, then there exists a signature matrix $S$ such that $\vec{D}_{\vec{G}}^1 = \vec{D}_{\vec{G}}^2 \cdot S$.

Observation 2.10. Let $D_{\vec{G}}$ be an incidence matrix for the strongly connected oriented graph $\vec{G}$. If $I = \{I_1, \ldots, I_k\}$ is a partition of the the columns of $D_{\vec{G}}$ which satisfies the columns condition, then $I_1$ is an edge-disjoint union of cycles.

Theorem 2.11. Let $D_{\vec{G}}$ be an incidence matrix for the strongly connected oriented graph $\vec{G}$. Any set $\{I_1, \ldots, I_t\}$ such that $I_i \in E_{\vec{G}}$, $I_j \cap I_l = \emptyset$ if $j \neq l$, and for all $1 \leq j \leq t$, $\sum_{i \in I_j} a_i \in \{a_i : i \in \bigcup_{l=1}^t I_l\}$, can be extended to a partition of $E_{\vec{G}}$ that satisfies the columns condition.

Corollary 2.12. Let $D_{\vec{G}}$ be an incidence matrix for the strongly connected oriented graph $\vec{G}$. The greedy algorithm produces a partition of the columns of $D_{\vec{G}}$ which satisfies the columns condition.
3 Sign Patterns

A sign pattern matrix (or sign pattern for short) is a (rectangular) matrix having entries in \{+,-,0\}. For a real matrix \(A\), \(\text{sgn}(A)\) is the sign pattern having entries that are the signs of the corresponding entries in \(A\). If \(\mathcal{Y}\) is an \(n \times n\) sign pattern, the sign pattern class (or qualitative class) of \(\mathcal{Y}\), denoted \(Q(\mathcal{Y})\), is the set of all \(A \in \mathbb{R}^{n \times n}\) such that \(\text{sgn}(A) = \mathcal{Y}\). It is traditional in the study of sign patterns to say that a sign pattern \(\mathcal{Y}\) requires property \(P\) if every matrix in \(Q(\mathcal{Y})\) has property \(P\) and to say that \(\mathcal{Y}\) allows property \(P\) if there exists a matrix in \(Q(\mathcal{Y})\) that has property \(P\). Patterns that require partition regularity are too trivial to be of interest:

**Theorem 3.1.** The only sign patterns that requires partition regularity are the all zero sign patterns.

**Proof.** Assume \(\mathcal{Y} = [\psi_{ij}]\) has a nonzero entry. Construct a matrix \(A = [a_{ij}]\) as follows:

- For all \(i,j\) such that \(\psi_{ij} = 0\), \(a_{ij} = 0\).
- For all \(i,j\) such that \(\psi_{ij} = +\), \(a_{ij} = 1\).
- For all \(i,j\) such that \(\psi_{ij} = -\), \(a_{ij} = -\frac{1}{n}\).

There is no subset of columns that sum to zero, so \(A\) does not have the columns condition and so is not partition regular. \(\square\)

Since a partition regular matrix must satisfy the columns condition, it is clear that in order for a sign pattern to allow partition regularity, any nonzero row must have both at least one + entry and at least one − entry. This is also sufficient for a sign pattern to allow partition regularity:

**Theorem 3.2.** Let \(\mathcal{Y}\) be an \(m \times n\) sign pattern. The following are equivalent:

1. Each row of \(\mathcal{Y}\) has at least one + entry and at least one − entry or every entry is 0.
2. \(\mathcal{Y}\) allows CC(1).
3. \(\mathcal{Y}\) allows partition regularity.

**Proof.** It is clear that \((2) \implies (3) \implies (1)\). Assume each row of \(\mathcal{Y} = [\psi_{ij}]\) has at least one + entry and at least one − entry or every entry is 0. If row \(i\) is non entirely 0, let \(m(i)\) denote the column number of the first − entry in row \(i\); otherwise, \(m(i) = 0\). Construct a matrix \(A = [a_{ij}]\) as follows:

- For all \(i,j\) such that \(\psi_{ij} = 0\), \(a_{ij} = 0\).
- For all \(i\) such that \(m(i) > 0\):
  - If \(\psi_{ij} = +\), \(a_{ij} = 1\).
  - For \(j > m(i)\), if \(\psi_{ij} = -\) then \(a_{ij} = -\frac{1}{n}\).
  - \(a_{i,m(i)} = -\sum_{j \neq m(i)} a_{ij}\).

Clearly \(A \in Q(\mathcal{Y})\) and \(A\mathbf{1} = \mathbf{0}\), so \(A\) has CC(1). \(\square\)

The minimum rank of an \(m \times n\) sign pattern \(\mathcal{Y}\) is

\[\text{mr}(\mathcal{Y}) = \min\{\text{rank}(A) : A \in Q(\mathcal{Y})\},\]

and the maximum nullity of \(\mathcal{Y}\) is

\[\text{M}(\mathcal{Y}) = \max\{\text{null}(A) : A \in Q(\mathcal{Y})\}.\]
Clearly $mr(Y) + M(Y) = n$.

It is not always the case that the nullity of a partition regular matrix can be realized by the number of cells in a columns condition partition. For example, for $A = \begin{bmatrix} 3 & -1 & -1 & -1 \end{bmatrix}$, $\text{null}(A) = 3$ but $A$ is $\text{CC}(m)$ only for $m = 1$. But the following question remains open:

**Question 3.3.** If $Y$ allows partition regularity, must there exist a matrix $A \in Q(Y)$ such that $A$ is $\text{CC}(M(Y))$?

**Theorem 3.4.** Let $G$ be a connected graph, let $\vec{G}$ be an orientation of $G$, and let $Y = \text{sgn}(D_{\vec{G}})$. Then the following are equivalent:

1. $Y$ allows PR
2. $D_{\vec{G}}$ is PR
3. $\vec{G}$ is strongly connected.

**Proof.** [will write this later] \hfill $\square$

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