VIII. Exponential and logarithmic functions

A. Exponential functions

Definition. The function \( f(x) = b^x \), where \( b \) is a positive constant, is called the exponential function with base \( b \). It is defined for all real numbers \( x \), but see note below.

To graph, we plot a few points and join them with a smooth curve.

Example: \( f(x) = 2^x \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>-1</th>
<th>-2</th>
<th>-3</th>
<th>1/2</th>
<th>3/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2^x )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>1/2</td>
<td>1/4</td>
<td>1/8</td>
<td>( \sqrt{2} \approx 1.4 )</td>
<td>(( \sqrt{2} ))^3 \approx 2.7</td>
</tr>
</tbody>
</table>

(The other graphs shown below were obtained similarly.)

Note: There is no easy way to compute (or even to define) such numbers as \( 2^\pi \). We can approximate them, however. The number \( \pi \) can be thought of as the limit of the sequence

\[ 3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots \]

Then we get \( 2^\pi \) as the limit of the sequence

\[ 2^3, 2^{31/10}, 2^{314/100}, 2^{3141/1000}, \ldots \], which is approximately 8.825.

Properties
1. \( b^x > 0 \) for every \( x \).
2. \( b^0 = 1 \) for every \( b \) (so the graph of \( f(x) = b^x \) always passes through \( (0,1) \)).
3. If \( b > 1 \), then \( b^x \) increases without bound as \( x \) tends toward infinity and tends toward zero as \( x \) tends toward negative infinity. If \( 0 < b < 1 \), then \( b^x \) approaches zero as \( x \) tends toward infinity and increases without bound as \( x \) approaches negative infinity. Thus the graph of \( b^x \) (for \( b \neq 1 \)) has the \( x \)-axis as a horizontal asymptote.

B. Logarithms

Notice from the graphs above that if \( b > 0 \) but \( b \neq 1 \) then for each positive number \( y \) there is exactly one number \( x \) for which \( b^x = y \). This number is called the logarithm of \( y \) base \( b \) or the base-\( b \) logarithm of \( y \) and is written \( \log_b y \). Thus, by definition, \( \log_b y \) is the exponent to which we must raise \( b \) in order to get \( y \). Saying \( x = \log_b y \) is equivalent to saying \( b^x = y \). Note that only positive numbers have logarithms!

\[
\begin{align*}
\text{Examples} & \quad \log_2 8 = 3 \text{ since } 2^3 = 8 & \log_2 1 = 0 \text{ since } 2^0 = 1 \\
& \quad \log_3 81 = 4 \text{ since } 3^4 = 81 & \log_2 2 = 1 \text{ since } 2^1 = 2 \\
& \quad \log_2 \frac{1}{4} = -2 \text{ since } 2^{-2} = \frac{1}{4} & \log_2 8 = \frac{3}{4} \text{ since } 16^{3/4} = 8
\end{align*}
\]

To repeat: \( \log_b y = x \) is equivalent to \( b^x = y \).

Example Find \( x \) if \( \log_5 x = 4 \).

\[
x = 5^4 = 625 \quad \text{by definition of logs.}
\]

To graph a logarithmic function, note that if \((c,d)\) is a point on the graph of \( b^x \), so that \( d = b^c \), then \((d,c)\) will be a point on the graph of \( \log_b x \), because \( c = \log_b d \). So the log. graph is the "inverse" of the exponential graph.

Example

\[
f(x) = \log_2 x
\]
Properties For every $b > 0$ ($b \neq 1$):
1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. $\log_b (b^x) = x$, where $x$ is any number or expression.
4. $b(\log_b x) = x$, where $x$ is any positive number or expression.

You may be most familiar with base-10 logarithms, e.g. $\log_{10}(0.1) = -1$ since $10^{-1} = 0.1$; $\log_{10} 1000 = 3$ since $10^3 = 1000$.

Exercises VIII. AB
Sketch the graphs of the following functions.
1. $f(x) = 3^x$
2. $g(x) = (1/3)^x$
3. $h(x) = \log_3 x$
4. Find: (a) $\log_3 27$ (b) $\log_2 \sqrt{2}$ (c) $\log_{10} 100$ (d) $\log_3 (1/27)$
5. Solve for $x$: (a) $\log_2 x = 3$ (b) $\log_5 (1/x) = 1$ (c) $2^x = 3$
6. Simplify: (a) $\log_{10} (10^x)$ (b) $5 \log_5 x^2$

C. Rules of computation for logarithms
Since logarithms are related to exponential functions, each of the rules for exponents gives rise to a corresponding rule for logarithms. Besides the facts that have already been listed, there are the following properties of logs:

\[
\log_b (xy) = \log_b x + \log_b y \quad \text{(this comes from } b^x b^y = b^{x+y})
\]
\[
\log_b \frac{x}{y} = \log_b x - \log_b y
\]

In particular, $\log_b \frac{1}{y} = -\log_b y$ (since $\log_b 1 = 0$)

\[
\log_b x^y = y \log_b x \quad \text{(this comes from } (b^x)^y = b^{xy})
\]

Examples 1. If $a = \log_{10} 2$ and $b = \log_{10} 3$, write $\log_{10} 6$ in terms of $a$ and $b$.

\[
\log_{10} 6 = \log_{10} (2 \cdot 3) = \log_{10} 2 + \log_{10} 3 = a + b.
\]
2. With \( a \) and \( b \) as in (1), write \( \log_{10}(0.6) \) in terms of \( a \) and \( b \).

\[
\log_{10} 0.6 = \log_{10}(6/10) = \log_{10} 6 - \log_{10} 10 = a + b - 1.
\]

3. Write \( \log_3(x-1) + 2 \log_3(x-2) - 3 \log_3(x-4) \) as a single logarithm.

\[
\log_3(x-1) + 2 \log_3(x-2) - 3 \log_3(x-4)
= \log_3(x-1) + \log_3(x-2)^2 - \log_3(x-4)^3
= \log_3[(x-1)(x-2)^2] - \log_3(x-4)^3 = \log_3 \left( \frac{(x-1)(x-2)^2}{(x-4)^3} \right).
\]

4. Find \( x \) if \( 10^{(\log_{10}x^2 + 3\log_{10} x)} = 2 \).

\[
\log_{10} x^2 + 3 \log_{10} x = \log_{10} x^2 + \log_{10} x^3
= \log_{10} (x^2 \cdot x^3) = \log_{10} x^5,
\]
so \( 10^{(\log_{10} x^2 + 3 \log_{10} x)} = 10^{\log_{10} x^5} = x^5 = 2 \),
and \( x = 2^{1/5} = 5^{\frac{1}{2}} \).

**Exercises VIII C**

1. Write as a single logarithm.
   (a) \( \log_b(x+1) + \log_b(x-2) + 2 \log_b(x-3) \)
   (b) \( \frac{1}{2} \log_b(x+1) - \frac{1}{2} \log_b(x-1) \)

2. Let \( a = \log_{10} 2 \), \( b = \log_{10} 3 \), \( c = \log_{10} 5 \). Write the following in terms of \( a \), \( b \), and \( c \):
   (a) \( \log_{10} 360 \)
   (b) \( \log_{10} \frac{54}{25} \)

3. Write using sums and differences of logs and only first powers of \( x \).
   (a) \( \log_b \frac{x+1}{x+2} \)
   (b) \( \log_b \frac{(x-1)^2(2x+1)^3}{\sqrt[3]{(4x-1)^2}} \)

4. Solve for \( x \):
   (a) \( \log_2 \sqrt{3x+1} = 1 \)
   (b) \( 3^{-2} \log_3 x = 1/3 \)
D. The natural logarithm

There is a special number, \( e \), equal to approximately 2.71828, which occurs frequently in mathematics and the sciences. The logarithm using \( e \) as base turns out to be most important. This logarithm is called the natural logarithm and one often writes \( \ln \) instead of \( \log_e \). Thus \( y = \ln x \) means \( y = \log_e x \) which means \( e^y = x \). Sometimes instead of \( e^x \) one writes \( \exp(x) \). This is called the natural exponential function. All the usual properties of exponents and logarithms hold for the functions \( \exp(x) \) and \( \ln x \).

Answers to Exercises VIII

AB:

1. \( y = 3^x \)
2. \( y = \frac{1}{3^x} \)
3. \( y = \log_3 x \)

4. (a) 3  (b) 1/2  (c) 2  (d) -3
5. (a) \( x = 2^3 = 8 \)  (b) \( x = 5 \)  (c) \( x = \log_2 3 \)
6. (a) \( x \)  (b) \( x^2 \)

C: 1. (a) \( \log_b (x+1)(x-2)(x-3)^2 \)  (b) \( \log_b \sqrt{\frac{x+1}{x-1}} \)
2. (a) \( \log_{10} 360 = \log_{10} (2^2 \cdot 3^2 \cdot 10) = 2\log_{10} 2 + 2\log_{10} 3 + \log_{10} 1 \)
   \[= 2a + 2b + 1\]
   (b) \( \log_{10} (54/25) = \log_{10} 54 - \log_{10} 25 = \log_{10} (2 \cdot 3^3) - \log_{10} 5^2 \)
   \[= a + 3b - 2c\]
3. (a) \( \log_b (x+1) - \log_b (x+2) \)
   (b) \( 2 \log_b (x-1) + 3 \log_b (2x+1) - (2/3) \log_b (4x-1) \)
4. (a) \( \frac{1}{2} \log_2 (3x+1) = 1 \), \( 2 \log_2 (3x+1) = 3x+1 = 2^2 \), so \( x = 1 \).
   \( \log_3 \frac{1}{x^2} = \frac{1}{2} = \frac{1}{3} \), so \( x = \pm \sqrt{3} \)