VI. Trigonometry, part 2, plus conic sections.

A. Trigonometric graphs

1. From definitions and data in Part 1 we can graph the trig. functions.

\[ f(\theta) = \sin \theta \]
\[ \text{period} = 2\pi \]
\[ \text{amplitude} = 1 \]

\[ f(\theta) = \cos \theta \]
\[ \text{period} = 2\pi \]
\[ \text{amplitude} = 1 \]

\[ f(\theta) = \tan \theta \]
\[ \text{period} = \pi \]

-- \( f(\theta) = \sec \theta \) is graphed on the next page, after Example 2 --

2. Variations on the basic graphs.

Example 1
\[ f(\theta) = 2\sin \theta \]
\[ \text{period} = 2\pi \]
\[ \text{amplitude} = 2 \]
(sine shown for comparison)
Example 2

\[ f(\theta) = \sin(3\theta) \]

period = \( \frac{2\pi}{3} \)

amplitude = 1

As \( \theta \) runs from 0 to \( 2\pi \), \( 3\theta \) runs from 0 to \( 6\pi \). Thus the \( \sin 3\theta \) curve oscillates 3 times as fast; its period is \( \frac{1}{3} \cdot 2\pi \). (The graph of \( \sin \theta \) is shown for comparison.)

\[ f(\theta) = \sec \theta \]

period = \( 2\pi \)

(The graph of \( \cos \theta \) is shown for comparison.)

B. Useful identities (Also see V C)

* \( \sin^2 \theta + \cos^2 \theta = 1 \) (i.e. \( (\sin \theta)^2 + (\cos \theta)^2 = 1 \)) \)

Pythagorean theorem

\[ \tan^2 \theta + 1 = \sec^2 \theta ; \quad 1 + \cot^2 \theta = \csc^2 \theta \]

* \( \sin(\theta \pm \xi) = \sin \theta \cos \xi \pm \cos \theta \sin \xi \)

* \( \cos(\theta \pm \xi) = \cos \theta \cos \xi \mp \sin \theta \sin \xi \)

\[ \sin 2\theta = 2 \sin \theta \cos \theta \]

\[ \cos 2\theta = \cos^2 \theta - \sin^2 \theta ; \quad \cos 2\theta = 1 - 2\sin^2 \theta ; \quad \cos 2\theta = 2\cos^2 \theta - 1 \]

\[ \sin^2 \theta = \frac{1 - \cos 2\theta}{2} ; \quad \cos^2 \theta = \frac{1 + \cos 2\theta}{2} \]

\[ \cos \theta = \sin(\theta + \pi/2) \]

Look at the graphs.

\[ \sin \theta = \cos(\theta - \pi/2) \]
\[
\cos \theta = \sin(\pi/2 - \theta)
\]
Think of
\[
\sin \theta = \cos(\pi/2 - \theta)
\]
right triangles.

\[
\begin{array}{c}
A \\ B \\ C \\
\end{array}
\]
Law of sines: \[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]
Law of cosines: \[
a^2 = b^2 + c^2 - 2bc \cos A
\]
(also \[b^2 = a^2 + c^2 - 2ac \cos B,\] etc.)

### Examples

1. Verify that for all \( \theta \),
\[
\frac{\cot \theta}{\csc \theta - 1} = \frac{1 + \sin \theta}{\cos \theta}.
\]
Soln: multiply left side by \( \frac{\sin \theta}{\sin \theta} \) to get \( \frac{\cos \theta}{1 - \sin \theta} \), then by \( \frac{1 + \sin \theta}{1 + \sin \theta} \) to get \( \frac{(1 + \sin \theta)\cos \theta}{1 - \sin^2 \theta} = \frac{(1 + \sin \theta)\cos \theta}{\cos^2 \theta} = \) right side.

2. Find \( \cos \frac{\pi}{12} \).
\[
\cos \frac{\pi}{12} = \cos \left( \frac{\pi}{3} - \frac{\pi}{4} \right) = \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4}
\]
\[
= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1 + \sqrt{3}}{2\sqrt{2}}.
\]

### Exercises VI LAB

1. Sketch the graph of \( f(\theta) = -\cos \theta \).
2. Sketch the graph of \( f(\theta) = 2 \sin 4\theta \). What is its period?
3. Verify that \( \tan \theta + \sec \theta = \frac{1 + \sin \theta}{\cos \theta} \).
4. Verify: \( \frac{1 + \tan^2 \theta}{\csc \theta} = \sec \theta \tan \theta \).
5. Express \( \cos^2 2a \) in terms of \( \cos 4a \).
6. Find \( \sin \frac{5\pi}{12} \) by using trig. identities.

C. Conic sections

1. Circles. (a) centered at origin with radius \( r \) : \( x^2 + y^2 = r^2 \).
   (b) centered at \((h,k)\) with radius \( r \) : \((x-h)^2 + (y-k)^2 = r^2\).

2. Parabolas.
   (a) Vertex at origin: \( y = kx^2 \) or \( x = ky^2 \). The larger \( |k| \) is, the narrower the parabola will be.

   (b) More general parabola will be \( Ax^2 + Bx + Cy + D = 0 \) or \( Ay^2 + By + Cx + D = 0 \) with \( A, C \neq 0 \). By completing the square, these may be rewritten as \( y = a(x-b)^2 + c \) or \( x = a(y-b)^2 + c \), which are parabolas with vertices at \((b,c)\) or \((c,b)\), respectively.
3. Ellipses.
   (a) Center at origin, x-intercepts ±a, y-intercepts ±b, a,b > 0:
   \[
   \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
   \]
   major semi-axis = a, minor semi-axis = b since a > b.

   (b) More general ellipse: \(Ax^2 + By^2 + Cx + Dy + E = 0\) with \(A,B \neq 0\), \(A, B\) same sign. Rewrite as
   \[
   \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1; \text{ center at } (h,k), \text{ semi-axes } a \text{ and } b.
   \]

4. Hyperbolas.
   (a) Center at origin, asymptotes \(ay = \pm bx\), a,b > 0:
   \[
   \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1
   \]

   (b) More general hyperbola: \(Ax^2 + By^2 + Cx + Dy + E = 0\), A,B = 0, A,B opposite signs.
   Rewrite as \(\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = \pm 1\), center at \((h,k)\).

   (c) \(xy = c\) is also a hyperbola; write as \(y = x/c\) to graph, for example,
Note: To display the equations of conic sections in a standard, recognizable form as above, it may be necessary to "complete the square". The technique comes from the formula \((x + a)^2 = x^2 + 2ax + a^2\); we can write \(x^2 + bx + c = (x + b/2)^2 + c - b^2/4\), getting rid of the \(x\) term. (In the exercises, this has already been done.)

Exercises VI.C
Identify and sketch the following conic sections.

1. \(8x = -3(y - (1/3))^2 + 4\)
2. \((x - 1)^2 + (y - 2)^2 = 9\)
3. \(9x^2 + 4y^2 = 36\)

Answers for Exercises VI

A: 1.

B: 3. Hint: write the left side in terms of \(\sin \theta\) and \(\cos \theta\).